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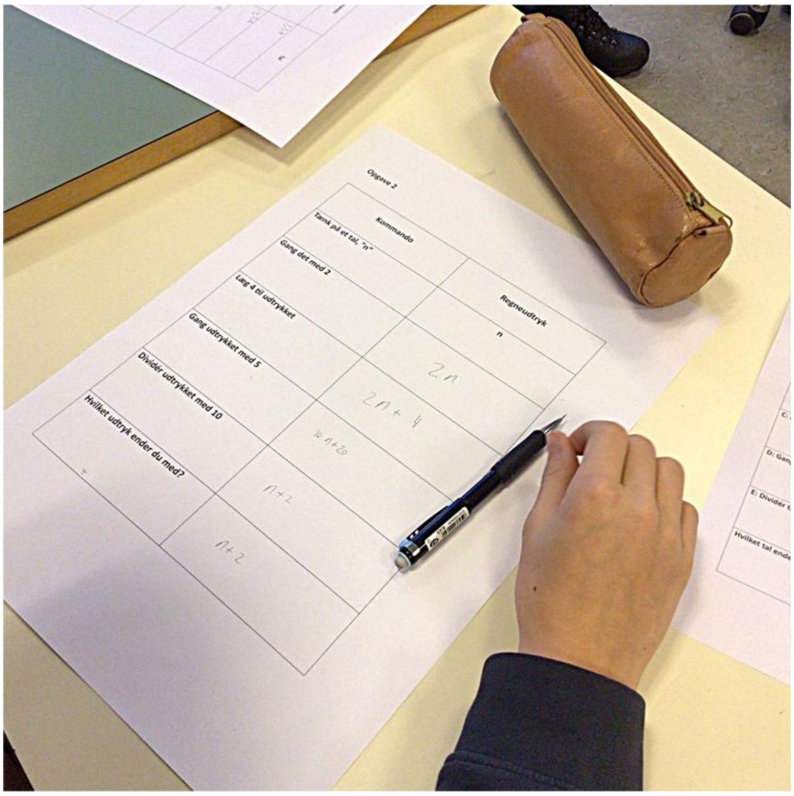

PhD Thesis

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Addressing Praxeological Differences in Algebra Through Bi-Institutional Lesson Study

Supervisor: Carl Winsløw

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Doctoral Dissertation 2026

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Cover image: Illustration of a student working on the task described in Paper 4.

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Abstract

The transition from lower secondary to upper secondary school in Denmark, particularly in mathematics, has long drawn attention from both society and the national and international research community. In 2022, a national expert group identified *Numbers and Algebra* as a key area of concern in this context. The PhD project has a twofold purpose.

First, the project investigates the transition problem in algebra. Based on the Anthropological Theory of the Didactic (ATD), the concept of praxeological differences is developed as a theoretical and methodological tool for identifying transition problems. This is based on a praxeological reference model for algebra at the secondary level, constructed through a praxeological analysis. Using this tool and the associated method, specific transition problems in algebra between lower secondary and upper secondary schools are identified. The results show that the transition problem is not primarily related to tasks involving the construction of algebraic expressions or substitution in such expressions. Instead, the transition problem is particularly related to the rewriting and transformation of algebraic expressions as well as to the use of algebraic techniques in solving first-degree equations.

Second, the project investigates how identified transition problems in algebra can be addressed through teacher professional development. Inspired by Japanese lesson study, the project experiments with what we call *bi-institutional lesson study*, in which teachers from two neighboring institutions (here, lower and upper secondary school in Denmark) participate. The project analyzes the establishment of the new paradidactic infrastructure and the conditions and constraints associated with its establishment in the specific Danish context. A main point in the project is to investigate how participation in bi-institutional lesson study can support the development of teachers' didactic practice and theoretical knowledge related to the praxeological differences.

It turns out that the establishment of a bi-institutional lesson study is strongly conditioned by the development of a shared model of the subject matter that is central to the transition problem. Such a model can function as a shared reference point across institutions. The teachers' mutual interest and engagement are also a central condition for the collaboration, while asymmetrical engagement constitutes a constraint. Furthermore, unclear and implicit roles and responsibilities constitute barriers to both the establishment of collaboration and the development or sharing of knowledge.

Finally, bi-institutional lesson study functions as an infrastructure in which praxeological differences in algebra are made explicit and addressed through joint task development, planning, observation, and reflection. The collaboration supports the development of teachers' didactic practice and theoretical knowledge related to algebraic rewriting and transformation. This includes knowledge of students' algebraic praxeologies, their difficulties with variables and algebraic expressions, as well as limitations of tasks that merely allow, but do not necessitate, algebraic rewriting. On the other hand, it is also clear that bi-institutional lesson study does not constitute a quick fix for transition problems and that its establishment and functioning depend strongly on facilitation and input from other institutions.

Resumé

Overgangen fra folkeskolen til gymnasiet i Danmark, særligt i matematik, har i en årrække været genstand for både samfundsmæssig opmærksomhed og forskningsmæssig interesse, både nationalt og international. I 2022 pegede en national ekspertgruppe på *Tal og Algebra* som et centralt område i denne overgang. Formålet med dette ph.d.-projekt er todelt.

For det første undersøger projektet overgangsproblematikken i algebra. Med afsæt i den Antropologiske Teori for Didaktik (ATD) udvikles begrebet *prakseologiske forskelle* (oprindeligt *praxeological differences*) som et teoretisk og metodisk greb til at identificere overgangsproblemer. Dette sker på baggrund af en praxeologisk referencemodel for algebra på sekundært niveau, konstrueret gennem en praxeologisk analyse. Ved hjælp af dette greb og den tilhørende metode identificeres konkrete overgangsproblemer i algebra mellem folkeskolen og gymnasiet. Resultaterne viser, at overgangsproblemet ikke primært knytter sig til opgaver, der involverer opstilling af algebraiske udtryk eller substitution i sådanne udtryk. Derimod peger resultaterne på, at overgangsproblemet især relaterer sig til omskrivning og transformering af algebraiske udtryk samt anvendelse af algebraiske teknikker i løsning af førstegradsligninger.

For det andet undersøger projektet, hvordan de identificerede overgangsproblemer i algebra kan adresseres gennem professionel udvikling for lærerne. Inspireret af japanske *lektionsstudier* eksperimenterer projektet med det, der betegnes som *bi-institutionelt lektionsstudie*, hvor lærere fra to naboinstitutioner (her folkeskolen og gymnasiet i Danmark) deltager. Projektet analyserer etableringen af den nye paradidaktiske infrastruktur samt de betingelser og begrænsninger, der knytter sig til dens etablering i den specifikke danske kontekst. Et centralt fokus er at undersøge, hvordan deltagelse i bi-institutionelt lektionsstudie kan understøtte udviklingen af lærernes didaktiske praksis og teoretiske viden relateret til de prakseologiske forskelle.

Det viser sig, at etableringen af et bi-institutionelt lektionsstudie i høj grad er betinget af udviklingen af en fælles model for det faglige indhold, som er centralt for overgangsproblemet (her algebra). En sådan model kan fungere som et fælles referencepunkt på tværs af institutionerne. Lærernes gensidige interesse og engagement er ligeledes en afgørende forudsætning for samarbejdet, mens asymmetrisk engagement udgør en begræsning. Desuden udgør uklare og implicitte roller og ansvarsfordelinger barrierer for både etableringen af samarbejdet og udvikling og deling af viden.

Afslutningsvis fungerer bi-institutionelt lektionsstudie som en infrastruktur, hvor prakseologiske forskelle i algebra gøres eksplicitte og bearbejdes gennem fælles opgaveudvikling, planlægning, observation og refleksion. Samarbejdet understøtter udviklingen af lærernes didaktiske praksis og teoretisk viden relateret til algebraisk omskrivning og transformation. Dette omfatter viden om elevers algebraiske praxeologier, deres vanskeligheder med variable og algebraiske udtryk samt begrænsninger ved opgaver, der blot muliggør, men ikke nødvendiggør, algebraisk omskrivning. Samtidig fremgår det tydeligt, at bi-institutionelt lektionsstudie ikke udgør en hurtig genvej til løsning af overgangsproblemer, og at både etablering og funktion i høj grad er afhængig af facilitering og input fra andre institutioner.

List of Papers

Paper 1

Cosan, D. D. (2024). Praxeological differences in institutional transition: The case of school algebra. *Annales de Didactique et des Sciences Cognitives, 29*, 217–238. https://doi.org/10.4000/12ym6

Paper 2

Cosan, D. D. (n.d.). Establishing bi-institutional lesson study for the transition from lower secondary school to upper secondary school in algebra. Manuscript submitted for publication. *Hiroshima Journal of Mathematics Education.*

Paper 3

Cosan, D. D. (n.d.). Teachers' development of didactic and algebraic praxeologies through bi-institutional lesson study. Manuscript submitted for publication. *Recherches En Didactique Des Mathématiques*.

Paper 4

Cosan, D. D., & Winsløw, C. (n.d.). Transition problems and lesson study in secondary school. Under revision for publication in *International Journal for Lesson and Learning Studies*.

Other Papers from the Ph.D. Period

Cosan, D. D. (2023). Improving middle school algebra through bi-institutional lesson study. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13),* (pp. 4906–4907). Alfréd Rényi Institute of Mathematics and ERME.

Cosan, D. D. (2025). Bi-institutional lesson study on algebra: What is it and what can lower secondary school teachers learn? In M. Bosch, G. Bolondi, S. Carreira, C. Spagnolo, & M. Gaidoschik (Eds.), *Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14)* (pp. 4473–4480). Free University of Bozen-Bolzano and ERME.

Cosan, D. D. (to appear). Institutional transition between lower and upper secondary: Using praxeological differences and lesson study to improve algebra teaching in Denmark. In F. Wozniak (Eds). *Proceedings of the Eighth International Conference on the Anthropological Theory of the Didactic (CITAD8)*, to appear. Centre de Recerca Matemàtica & ERME.

Table of Contents

1 Introduction

The idea and interest for this project can be traced back to 2019, when I started working as a mathematics teacher at a Danish upper secondary school. I received students who had just completed lower secondary school, and these students, with varying levels of prior knowledge, shared a common challenge: algebra. Some of them had almost no knowledge of what an equation was, while others struggled with solving so-called complicated equations, such as those with an x on both sides of the equal sign. Even students who reported having achieved high results in the final examination of lower secondary school arrived at upper secondary school with algebraic difficulties and a lack of knowledge.

These difficulties with algebra persist for many students, both in my own teaching experience and nationally. For example, Grønbæk et al. (2019) point out that students who completed two years of mathematics in upper secondary school still experienced difficulties with algebraic rewriting, as only 25% of the students were able to reduce the expression $(a+b)^2 - b \cdot (2a+b)$ to a^2.

Furthermore, a study by Mathiasen (2009) shows that Danish students experience the transition from lower secondary to upper secondary school as particularly difficult in mathematics compared with the subjects Danish and English.

Like many other upper secondary school teachers, I initially had little knowledge of how mathematics was taught in lower secondary school, what challenges students face, or what kind of mathematical content they worked with. This changed when I worked on my master's thesis (Cosan, 2021), which allowed me to investigate mathematics, and in particular arithmetic and algebra, in grades 5 and 7. I examined textbooks, curriculum, and the official program for lower secondary school to get knowledge about how arithmetic and early algebra were presented at these levels. I then designed a diagnostic test to identify the lack of knowledge and misconceptions in students' knowledge of arithmetic and early algebra.

Writing this master's thesis sparked a growing interest in lower secondary mathematics, concretely, algebra. My own students' difficulties, the national problem with algebra, and my emerging interest in lower secondary school together motivated me, when the opportunity arose, to focus this project on the transition from lower secondary to upper secondary school in algebra.

Throughout their educational life, students experience many transitions, such as from pre-school to primary school, from primary to secondary school, and from lower secondary to higher secondary school. In these transitions, many changes occur: new buildings, a new school, the number of teachers and students, new timetables, and so on (Gueudet, 2016).

As formulated in Gueudet (2016), such transitions may also involve changes "in the kind of knowledge and know-how that is taught, as well as in the specific didactic activities related to this knowledge and know-how" (p. 16). In this project, transitions between institutions are therefore understood, as described in Gueudet (2016), as "a change at the level of school and in relation to the teaching and learning of a specific discipline or field of knowledge" (p. 17).

De Vleeschouwer (2010) investigated the transition between secondary and tertiary level and pointed out that the difficulties in transition "can also be caused by the possibility that the same mathematical notion will be approached differently in the secondary school institution and in the undergraduate institution" (p. 155). Although this study addresses the secondary-tertiary transition, the same concern can apply to other transitions, such as the transition between lower secondary and upper secondary school. Only limited research has examined transitions between institutions with a focus on the discipline, and specifically on the mathematical domain, with the exception of Carraher and Schliemann's (2014, as cited in Gueudet, 2016) research on early algebra. Algebra, however, "has long been the "transition topic" par excellence, marking the frontier between elementary and secondary education" (Gueudet, 2016, p. 18).

Although much research on transitions focuses on factors such as pedagogy, school, and society, it is important to consider specific mathematical content, as this is the medium through which student-teacher relationships are established and teaching and learning become concrete (Gueudet, 2016, p. 21). As proposed by Gueudet (2008), such transition problems can be studied by focusing on the mathematical content on different levels.

Gueudet (2016) also points out that many initiatives aiming to smooth the transition to secondary school and to increase students' engagement in mathematics often focus on strengthening the relationships between primary and secondary teachers (p. 18). This approach motivates this project's focus on teacher collaboration and teacher professional development.

This project, therefore, has two main objectives:

1. To present a general methodology for identifying transition problems and, based on this, to investigate the transition problem in algebra from Danish lower secondary to upper secondary school
2. To experiment with a lesson study format, inspired by Japanese teacher professional development, between lower secondary and upper secondary school, and to examine how such collaboration can help address the transition problem.

1.1 Navigating the thesis

The thesis is structured in the following way. Chapter 2 presents the theoretical framework guiding the study, the Anthropological Theory of the Didactic, and introduces central concepts relevant to the PhD project. Chapter 3 outlines the methodology used for the two literature reviews: one focusing on lesson study and one focusing on arithmetic and algebra. Chapter 4 then presents the literature on lesson study in Japan and beyond. Chapter 5 addresses transitions and algebra, with particular emphasis on the transition from arithmetic to algebra. Chapter 6 presents the context of the study, including the transition problem related to algebra in the Danish context and the participating teachers' different educational backgrounds. Building on these chapters, Chapter 7 presents the research questions of the PhD project, while Chapter 8 outlines the methodology used to address each research question. Chapter 9 presents and discusses the findings related to each research question, based on the four papers that constitute the PhD project. Finally, Chapter 10 discusses the results and outlines further perspectives and implications of the study, and Chapter 11 concludes the PhD project.

2 The Anthropological Theory of the Didactic

The Anthropological Theory of the Didactic (ATD) is a research framework in didactics of mathematics developed by Yves Chevallard in the 1980s (Bosch et al., 2020, p. xii). In ATD, "didactic" refers to any activity aimed at helping someone study or learn something (Chevallard & Sensevy, 2014). Didactic phenomena are understood as a fundamental part of being human: to be human involves co-creating and disseminating knowledge, as well as sometimes failing in this process (Bosch et al., 2020). Such phenomena encompass any situation in which someone intends to enable another person to know or do something, and they are not only found in school institutions but also in places such as television, exhibits, and sport fields (Mortensen & Winsløw, 2011).

The aim of ATD is not only to describe teaching and learning situations, but also to explain and question them by examining and comparing the observed practices with those that could have happened but did not (Bosch & Gascón, 2014). It also seeks "analyzing the conditions that enable teaching and learning processes to happen in the way they happen, while hindering or impeding other kinds of activities from taking place" (Bosch & Gascón, 2014, p. 80). As emphasized in Chevallard (2022), "what happens in the classroom can depend on conditions formed outside the classroom, sometimes very far away, in social space and historical time, of students and teachers" (p. 6)

Gascón (2024) points out that in ATD, "didactic phenomena refer not only to the *dissemination,* but also to the *genesis* and *development* of mathematics, which establishes that "the mathematical" and "the didactic" are, in a certain sense, inseparable" (p. 1321).

Although ATD originated in mathematics education, it has evolved to explain and understand the teaching and learning of any kind of knowledge across different domains (Chevallard, 2019). By adopting an anthropological perspective, it seeks to study the didactic phenomena where it appears, focusing on how knowledge is institutionally constructed and on the conditions that facilitate its dissemination (Chevallard & Bosch, 2020b).

2.1 Institutions in terms of ATD

Institutions play an important role in ATD in relation to mathematical and didactic activities. As Bosch & Gascón (2014) note, "doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, are considered as human activities taking place in institutional settings" (p. 68). An institution, I, understood broadly, is anything created or instituted that people can be members of (permanent or temporary), such as a class, a family, a football team, or a research group. Every institution contains institutional positions (Chevallard & Bosch, 2020a, p. xxxi; Chevallard & Bosch, 2020b, p. 54). As Chevallard (2019) points out, knowledge can either be created, used, or taught in an institution. A person is any human being, such as a newborn, an infant, a toddler, or a student, and the relation between persons and institutions is linked to the notion of position, p: a person who is a member of an institution occupies a position within it.

In terms of ATD, this is expressed as x is a *subject* of I in position p (Chevallard & Bosch, 2020b, p. 54; Chevallard, 2019). For example, a class as an institution includes both student and teacher positions.

Beyond the notion of person, institution, and positions, ATD also introduces the notion of object o, which can be an entity recognized as existing by at least one person. This can be words and concepts, mathematical notions, or techniques (Chevallard & Bosch, 2020b, p. 54)

These four notions together form a unit $R(x,o)$ which refers to the personal relation of x to o. This unit contains all how x interacts with, thinks about, or handles o. This includes considering, using, discussing, imagining, or even dreaming about the object. In this sense, the personal relation $R(x, o)$ represents what x "knows" about o: if $R(x, o)$ is non-empty, we say that x knows o. The content of $R(x, o)$ provides a formal way to think about the knowledge a person in an institution with a position has about a particular object (Chevallard, 2019).

2.2 Praxeology

According to ATD, any human knowledge, mathematical or otherwise, can be modelled in terms of praxeologies (Gascón, 2024). A praxeology consists of a praxis block and a logos block, each with two components. The praxis block, the "know-how" part of knowledge, consists of types of task T (e.g., "to work out a multiplication", "to cook an omelet", "to establish the algebraic expression of the area of a regular heptagon") and corresponding techniques, τ, which refers to "way of doing" a task of type T (Chevallard, 2019, p. 83).

The logos block contains the components of technology and theory. Technology (θ) refers to the explicit knowledge about the praxis, including a discourse used to describe, explain, and justify the techniques. Theory, denoted as Θ, refers to a more abstract and general discourse supposed to justify the technology (Chevallard, 2007). Chevallard (2019) points out that "such a discourse can vary from institution to institution, and even from position to position within a given institution" (p. 87).

Any human practice can be modelled in terms of praxeologies. In mathematics education, the praxis of a mathematical praxeology consists of a mathematical type of task, such as solving a linear equation, and the corresponding techniques for solving this task (for instance, using the distributive property). The logos contains the explanations, descriptions, and justification of the techniques and technology, drawing on mathematical definitions of terms, rules, theorems, proofs, etc. (Miyakawa & Winsløw, 2019). Learning mathematics can thus be seen as a change in the students' praxeological equipment, i.e., the praxeologies available to them in a given context (Gascón, 2024). The praxis and logos together form a mathematical praxeological organization, also called mathematical organization or mathematical praxeology (Barbé et al., 2005).

In the teaching of mathematical praxeologies, didactic praxeologies are also required. These include tasks and techniques for teaching, such as how to organize a classroom discussion, how to present new mathematical content, and how to guide students in their work with a mathematical task. Didactic praxeologies can be more generic at the theoretical level, drawing on more pedagogical content in the justification of the technology (Miyakawa & Winsløw, 2019). They are crucial for "making other praxeologies start living in and migrating within human groups" (Bosch & Gascón, 2014, p. 69). While students' knowledge consists of mathematical knowledge, mathematics teachers' core knowledge consists of didactic praxeologies. These praxeologies cannot be considered independent of the mathematical praxeology, since it is precisely these praxeologies that the didactic praxeologies aim to illuminate and work with, making them inseparable (Miyakawa & Winsløw, 2019).

Within ATD, Paper 1 introduces the concept of praxeological difference as a theoretical and methodological tool for analyzing and identifying transition problems between institutions. It refers to the difference between the mathematical organization students are expected to have mastered when entering a new institution I_2, denoted MO^{I_2}, and the mathematical organization they actually acquired at their previous institution I_1, denoted as MO^{I_1}. The praxeological difference is formally written as $MO^{I_2} \setminus MO^{I_1}$, and can appear at different levels of a praxeology, such as the type of task, the techniques, the technology, or the theory. This concept enables a precise identification of the knowledge gaps that may lead to difficulties during institutional transition.

2.3 Reference epistemological model

Mathematics education research addresses central questions, such as: What is a mathematical domain (for example, algebra, geometry, or statistics)? What kind of praxeologies is it made of? How is it interpreted in a given institution? What is it for? How is it related to other domains? ATD proposes Reference Epistemological Models (REMs) to address such questions (Bosch, 2015; Ruiz-Munzón et al., 2013), and when the model is constructed in terms of praxeologies, it is called a Praxeological Reference Model (PRM) (Bosch, 2015). It is not just a description of mathematical content but a framework that guides research. It represents an epistemological viewpoint that is assumed at the start (a priori), and which can evolve and be questioned over time. This viewpoint shapes: (1) the scope of the mathematical domain under study, (2) the didactic phenomena that are "visible" to researchers, and (3) the explanations and actions that are considered "acceptable" within a field of research (Bosch, 2015, p. 61).

The analysis of mathematical praxeologies is typically based on curricular documents and resources, textbooks, and teaching material (Artigue, 2022, p. 274). As emphasized by Barbé et al. (2005), for the Spanish case, these official programs and textbooks may offer "a set of mathematical elements (types of problems, techniques, notions, properties, results, etc.) that constitutes the knowledge *to be taught*" (pp. 240-241). REMs can be used to describe and analyze which mathematical praxeologies are taught and learnt in different institutions, what is left out, and why. They can also show how these choices affect other areas of mathematics (Ruiz-Munzón et al., 2013, p. 2875).

To construct a REM for domains such as geometry or algebra, a praxeological analysis must be conducted. This analysis results in a praxeological reference model containing "concrete activities that can be considered as the raison d'être of the mathematical content involved in terms of problems to be solved or questions to be addressed, as well as the way it takes form and evolves to give rise to new problematic questions" (Ruiz-Munzón et al., 2013, p. 2872). Wijayanti & Winsløw (2017) point out that a praxeological reference model "is not constructed independently from the material to be analyzed, but it is constructed *along with the analysis* and serves, in the end, to make that analysis completely explicit" (p. 315). A praxeological reference model should also be reproducible, so that other researchers using the model will come up with the same analysis (p. 315).

Gascón (2024) notes that the praxeological analysis often begins when researchers observe a remarkable and surprising didactic phenomenon that requires explanation (p. 1323). A REM then acts as a tool to represent the knowledge and practices involved (Gascón, 2024).

Several studies illustrate the use of praxeological reference models.

Wijayanti & Winsløw (2017) present a new approach to textbook analysis, where they construct a praxeological reference model for the sector of *proportion*, where they define explicit types of tasks and corresponding techniques, which allow them to analyze and compare three Indonesian textbooks. One of the key points in Wijayanti & Winsløw (2017) is that a praxeological reference model "enables us to analyze the mathematical core of textbooks in a quite objective and detailed way, which could contribute to "common measures" for both comparative and historical studies of how a sector or theme appears in mathematics textbooks" (p. 327).

Putra (2018) developed a praxeological reference model for rational numbers and used it to study Indonesian teachers' mathematical and didactic knowledge of rational numbers. Aoki (2023) constructed a praxeological reference model for arithmetic of fractions in Japanese primary schools to investigate how the content related to arithmetic of fractions is implemented in Japanese supplementary schools in Denmark.

Cosan (2021) develops a praxeological reference model for arithmetic and algebra in Danish fifth- and seventh-grade, and based on this model, she constructs a diagnostic test to examine whether students master these praxeologies.

2.4 Transitions in the sense of ATD

Artigue (2017) discusses transition issues in mathematics education, particularly the transition from secondary education to university, and highlights why ATD is well-suited for studying such transitions. Mathematical knowledge is always practiced within an institutional context, so "knowing" a concept has different meanings depending on one's institutional position, such as student or teacher. The notion of praxeology makes it possible to examine these differences of institutional relationships at the level of praxis and logos, and ATD emphasizes the conditions and constraints shaping what can be taught and learned in a given institution (p. 405).

Putra (2019) introduces the notion of praxeological change to describe the practical and theoretical change from natural to rational numbers. A praxeological change occurs when existing mathematical practices and theories cannot be extended to a new area. More generally, it involves "an institutional deconstruction and reconstruction of a set of new praxeologies in order to cope with an extended set of objects or tasks, when the old praxeologies only apply to a part of these and cannot be extended or generalised to apply to them all." (p. 3). Although

Putra (2019) focuses on natural and rational numbers, the concept is also relevant to transitions such as from arithmetic to algebra (Tonnesen, 2025).

2.5 Didactic and paradidactic infrastructure

Didactic infrastructure in ATD refers to the conditions and constraints that affect and shape the use of and development of teachers' mathematical and didactic praxeologies in teaching (Miyakawa &Winsløw, 2019). Chevallard (2019) describes infrastructure as "the underlying base needed to develop any determined reality" (p. 84).

This underlying base consists of the conditions and constraints that make the teachers' development possible. These depend on an institutional system, which can include more general conditions and constraints applicable to all teachers (e.g., a school or municipality), as well as specific conditions and constraints directly related to the teaching of mathematics and students' work with the mathematical praxeologies (Miyakawa & Winsløw, 2019).

Teachers' activities outside the classroom, such as lesson preparation, attending meetings, or participating in courses, that contribute to their development of mathematical and didactic praxeologies, are defined as paradidactic activities. While these activities are not part of actual teaching, they are closely related to it (Miyakawa & Winsløw, 2019). Like teachers' didactic activities in a teaching situation, their paradidactic activities are shaped by the conditions and constraints under which teachers work outside of teaching. Within the ATD framework, the paradidactic infrastructure consists of these conditions and constraints (Miyakawa & Winsløw, 2019). As described by Miyakawa and Winsløw (2019): "a paradidactic infrastructure may also involve institutionally given frameworks such as the examples of lesson study" (p. 284). Lesson study continues to be an active object of research within ATD (e.g., García et al., 2022; García & Lendínez, to appear; Mizoguchi et al., to appear; Strømskag & Cabero, to appear).

A bi-institutional lesson study is an element of paradidactic infrastructure proposed by us (Paper 2, Paper 3). It extends the idea of lesson study within one institution by focusing on transition problems between these two connected and neighboring institutions, referred to as I_1 and I_2. In a bi-institutional lesson study, teachers from I_1 and I_2 together plan, conduct, observe, and reflect on a lesson in I_1 to facilitate students' transition from I_1 to I_2 in view of concrete and urgent transition problems. These problems stem from praxeological differences between the two institutions and are known to cause difficulties for students moving across them. The collaboration is asymmetrical, as teachers from I_1 can be expected to have in-depth knowledge of the students' current mathematical praxeologies, while teachers from I_2 can be expected to be familiar with the mathematical praxeologies expected when entering I_2. A bi-institutional lesson study can thus be seen as a paradidactic infrastructure, as it provides an institutionally given framework for teachers to work together outside of teaching to study and develop didactic praxeologies in view of praxeological differences that lead to the transition problems between the two institutions.

3 Methodology for Literature Review

3.1 Literature Review on Lesson Study

The literature on lesson study is extensive and diverse. This thesis does not aim to cover the entire field, but to delimit and examine the part that is (in a sense to be specified) most relevant to the project's focus. The review aims to:

1. Map how lesson study is described and practiced in Japan, including the role of institutional frameworks and paradidactic infrastructure.
2. Map how lesson study is described and adapted outside Japan, with attention to interpretation, organization, and institutional contexts.

Search strategy

The review began with an attempt to identify the most important scientific books published on lesson study. Our hypothesis was that this could provide a good basis for identifying the main overall trends in research on the subject. Journal articles will be incorporated at a later stage to explore details identified in this first step. The review is limited to English-language literature. This unfortunately excludes relevant lesson study research published in other languages, especially in Japanese.

Searches were conducted in Scopus and Springer, both of which include monographs and edited volumes within educational research. The search string used was:

"lesson study" OR "jugyou kenkyuu"

The search produced 34 unique titles (after removing duplicates), of which two were excluded due to lack of access to the full texts. The remaining 32 books were classified either as

1. Practice-oriented: handbooks or manuals aimed at guiding teachers through lesson study steps (for instance, how to plan the research lesson, how to write the learning goals for the lesson, etc.), typically without institutional, cultural, or historical focus.
2. Research-oriented: scientific books and edited volumes offering an institutional, organizational, cultural, or historical perspective of lesson study.

Based on these definitions, eight practice-oriented books were excluded (e.g., Cerbin, 2011; Placa, 2023), leaving 24 research-oriented books for detailed screening.

The remaining books were reviewed using titles, abstracts, introductions, and selected chapters. Screening was guided by the focus on lesson study as an institutionally embedded practice based on the following inclusion and exclusion criteria:

Books were included if they were

- Offering an institutional, organizational, cultural, or historical perspective on lesson study
- Discussing paradidactic infrastructure or teacher professional cultures

- Examining adaptations of lesson study outside Japan with explicit attention to the institutional contexts
- Contributing to understanding the nature, development, or transfer of lesson study

Books were excluded if they were:

- Manuals or guides on “how to carry out lesson study” aimed at practitioners.
- using lesson study merely as an evaluation tool (e.g., to study teacher learning or metacognition) without analyzing lesson study itself.
- focusing primarily on lesson study as a practical or methodological tool for task design or classroom implementation, without addressing institutional, cultural, or organizational contexts.

Six additional books were excluded based on these criteria (e.g., Kawauchi & Yanagimoto, 2012; Ikeda et al., 2025).

This resulted in a final corpus of 13 books, which formed the foundation of the review.

Synthesis approach

The review takes a synthesis-oriented approach rather than a purely descriptive one. Instead of summarizing each source separately, books are grouped based on their contribution—such as documenting Japanese practice, conducting comparative analysis, offering theoretical reflections, or examining Western adaptations. The goal is to provide a clear overview of how lesson study is understood and contextualized across different settings, and how institutional structures influence these practices.

The 13 selected books, including seminal works such as Fernandez & Yoshida (2004) and Dudley (2014), form the basis of the review. These are complemented by peer-reviewed articles by prominent lesson study researchers such as Catherine Lewis, Toshiakira Fujii, and Akihiko Takahasi, who are highly cited in the 13 books, as well as in a more recent literature review on lesson study by Seleznyov (2018).

Review of Journal and journal articles

The next step involved examining three journals that hold a significant position within mathematics education research in relation to lesson study and, more broadly, teacher professional development. While *The International Journal for Lesson and Learning Studies* (IJLLS) focuses primarily on lesson study, *The Journal of Mathematics Teacher Education* (JMTE) specializes in teacher education and teacher professional development. *ZDM – Mathematics Education* was included as it also features several key contributions to research on (and with) lesson study.

A search using the keyword “lesson study” was conducted in *ZDM* and *JMTE,* yielding the following results:

- *ZDM:* 128 articles
- *JMTE* 104 articles

For *IJLLS,* the author manually reviewed all 329 articles published since the journal began to appear in 2011.

The subsequent step involved reading the abstracts of all identified articles to determine the ones that substantially address issues related to transitions or interactions between neighboring educational institutions in relation to lesson study.

3.2 Literature Review on Arithmetic and Algebra

The purpose of this literature review was to provide an overview of how the transition from arithmetic to algebra at the secondary level has been addressed in research. The aim was not to conduct a comprehensive review, but rather to map key positions, authors, and contributions that have shaped research on this transition.

Searches were carried out in Google Scholar, Scopus, and ERIC using the following keyword combinations:

Search terms	Google Scholar	Scopus	ERIC
"arithmetic" AND "algebra" AND "transition"	102.000	144	92
"arithmetic to algebra" AND "transition"	3430	36	50
"arithmetic to algebra" AND "transition" AND "secondary"	2100	9	29

Compared to Scopus and ERIC, Google Scholar provides access to a wider range of sources in didactics and related areas. This broader coverage is relevant in a review of this kind, as it increases the possibility of capturing both central research contributions and also other texts, like popularising works, that have informed the field over time. Although Scopus and ERIC provide more manageable sets of results, and ERIC is specialized in educational research, Google Scholar was used as the main source because it includes numerous influential and frequently cited publications not found in the other databases, including classical research such as Kieran (2004), Filloy & Rojano (1989), Herscovics & Linchevski (1994), and Kaput (2008).

The Google Scholar search returned 2100 results ranked according to the relevance for the search terms. The 1000 highest-ranked results were reviewed in an attempt to identify influential authors and contributions, which may be overlooked if relying solely on SCOPUS and ERIC.

Early in the process, García Fajardo et al.’s (2025) bibliometric review (2003-2023) emerged, offering an independent mapping of research on the arithmetic-algebra transition. It was used to validate that the authors identified through my searches, e.g., Kieran, Kaput, Herscovics & Linchevski, are recognized as central in the field. Additional works (e.g., Kieran, 2022; Usiskin, 1988) were included when referenced both in this bibliometric review and in the rest of the literature identified by us.

No systematic snowballing was conducted, as the review did not aim for exhaustive coverage. Instead, selection was based on citation frequency and thematic relevance. Inclusion was based on whether a study explicitly addressed arithmetic and algebra, examined the transition between them, and focused on secondary students or the transition from primary to secondary school.

3.2.1 School Algebra within ATD Literature Review

To examine research on school algebra from the perspective of the Anthropological Theory of the Didactic (ATD), an initial search was conducted in Google Scholar using:

"arithmetic" AND "school algebra" AND "ATD"

The search returned 130 results, which were screened by title and abstract. Selected articles focused on school algebra within an ATD framework and contributed to understanding teaching and learning in this domain. Included articles: focused on the praxeological reference model or various approaches to school algebra within ATD. Excluded articles: related to the algebra-calculus transition, upper secondary school, or university-level didactics (e.g., Klein double discontinuity). Additional references (e.g., Bolea, 2004) were added through snowballing. This search provided a focused ATD perspective that supplemented the review of the arithmetic-algebra transition.

4 Literature review – Lesson Study

Rappleye and Komatsu (2017), as noted by Cheng (2019), employ the metaphor of a plant and its soil to illustrate the challenges of transplanting Lesson Study from Japan to other school systems in various countries. While the plant itself can be moved, it will not yield the same fruits if removed from the soil that originally nourished it — namely, lesson study's long history in Japan and the well-established institutional structure and practice that have supported its development over time. This metaphor underscores that lesson study is not merely a teaching method but part of a broader school system.

Building on this understanding, the present literature review examines the institutional and paradidactic frameworks within which lesson study operates in Japan, and explores how they underpin its success. Furthermore, it investigates how lesson study, when transplanted beyond Japan, has been adapted, simplified, and recontextualized to fit different institutional frameworks.

4.1 Lesson Study in Japan

Lesson study (in Japanese: *jugyou kenkyuu*) is a central part of Japanese teachers' professional development. It has been practiced for over a hundred years to examine and improve teaching practice through a systematic focus on students' learning and teachers' didactic methods (Tanaka, 2017). Broadly, lesson study can be described as a process in which teachers collaboratively study curriculum content and teaching materials, plan, conduct, and observe a lesson (called a research lesson), and reflect upon a lesson to improve both their own practice and their knowledge about how students learn (Fernandez & Yoshida, 2004; Lewis, 2016).

Lesson study can take place at different levels, but the most common format is the school-based one, known as *konaikenshu* ("in-school training"), which is well established in Japanese elementary, lower secondary, and junior high schools (Fernandez & Yoshida, 2004).

Although lesson study has a long history, the formalized school-based practice of *konaikenshu* first appeared in the early 1960s, and by the mid-decade, the combination of lesson study and *konaikenshu* was already widespread as an effective approach for improving teaching across schools (Fernandez & Yoshida, 2004, p.15). During the 1970s, the Japanese government began supporting the practice financially, which meant that *konaikenshu* became an integral part of national school development efforts.

Lesson study in Japan is not merely a method, but part of an institutional system of professional learning supported by both national policy and local school organization (Baba et al., 2018, p. 35). According to the Ministry of Education, Culture, Sports, Science and Technology (MEXT) in Japan, teachers are legally required to engage continuously in professional development, which can be materialized through *konaikenshu* and a range of formal training programs at national, prefectural, and municipal levels (Baba et al., 2018) This culture, in which teachers learn freely from one another, has contributed to the outstanding quality of Japanese education (Ishii, 2017, p. 57).

Although *konaikenshu* has official support, it still relies on the commitment of schools and teachers, and it is experienced in practice as a "semi-obligatory" part of school development, where participation is expected as part of teachers' collective professional responsibility (Fernandez & Yoshida, 2004). Lesson study is time-consuming but offers teachers opportunities to identify their strengths and weaknesses and to develop concrete competencies that directly improve teaching (Fernandez & Yoshida, 2004, p. 16).

To support this work, many Japanese schools establish a *konaikenshu* promotional committee (*kenshu-soshiki*), which plans, coordinates, and sustains the school's lesson study activities (Fernandez and Yoshida, 2004, p. 13). The committee typically consists of committed teachers, ensuring that lesson study remains teacher-led and practice-oriented. At the same time, school leadership, especially principals and vice-principals, plays a crucial role in providing organizational and temporal structures that enable the work. In this way, lesson study becomes part of the school's overall pedagogical development strategy (Fernandez & Yoshida, 2004).

As emphasized in Chapter 2.5, from an ATD perspective, lesson study can be labeled as a paradidactic infrastructure (Miyakawa & Winsløw, 2019), where teachers collectively work on the objects and aims of teaching outside the classroom context. The school provides time, space, and social structures, making lesson study an institutionally organized arena where the school's didactic practice is continuously negotiated and developed.

For Japanese teachers, "lesson study is a well-known and expected routine" (Lewis, 2016, p. 576), and teachers engage in lesson study in many different ways depending on their career stage and professional community. Preservice teachers often encounter lesson study during practicum, working with university instructors and mentor teachers to plan and conduct lessons that are later analyzed collectively. For novice teachers, lesson study is a key part of professional development, as they plan and implement lessons with mentors for shared observations and reflection (Fernandez & Yoshida, 2004). These practices are embedded in a system connecting teacher education institutions, schools, and municipal authorities in a dense network supporting teachers' continuing development (Baba et al., 2018).

Fernandez and Yoshida (2004) emphasize that *konaikenshu* differs from many other forms of teacher professional development because the entire school comes together around a long-term shared goal, *the konaikenshu goal*, which is regarded as crucial for students' learning and the school's instructional improvement (p. 10). These goals rarely concern specific subject content but rather broader learning dispositions, such as motivation, collaboration, and curiosity. Schools often work on the same overall goal for several years, adjusting focus annually to explore new aspects. Lesson study thus becomes the main practical tool for pursuing the *konaikenshu goal* (Fernandez & Yoshida, 2004).

Typically, *konaikenshu* follows cycles in which the school selects a shared focus (e.g., "how can we promote students' critical thinking?"), plans research lessons, observes them, and conducts post-lesson discussions that analyze student learning in detail. The post-lesson discussions often involve an external expert, a so-called *knowledgeable other,* and collaboration with such commentators is widely recognized as central to lesson study. These commentators, university researchers, or experienced teachers, observe research lessons and help connect teachers' concrete observations to didactic theory during the post-lesson discussions (Murata, 2011, p. 5).

From an ATD perspective, *knowledgeable others* act as carriers of both didactic and mathematical knowledge, enabling teachers' concrete observations to be generalized into didactic or mathematical praxeologies, which is why they are expected to have strong didactic and mathematical knowledge.

Takahashi (2014) points out that educators outside Japan who show an interest in knowledgeable others have raised questions about "who these experts are, what their preparations are, what they talk about in their comments, and what they try to accomplish" (p. 3). To illustrate this, Takahashi (2014) summarizes findings from Fernandez et al. (2001), who identify three central reasons for involving knowledgeable others in research lessons: "to provide a different perspective on the lesson study work of the group; to provide information about the subject matter content, new ideas, or reforms; and to share the work of other lesson study groups" (cited in Takahashi, 2014, p. 3). Takahashi's (2014) study shows that in Japan, the role of the knowledgeable other has developed through participation in lesson study rather than through formal training. The knowledgeable others he interviewed "did not receive any formal training to become final commentators" and emphasized that "the best way to understand the role of the knowledgeable other is through participating in lesson study with colleagues" (Takahashi, 2014, p. 10). Their expertise is thus developed through "years of observing many research lessons and final comments by colleagues and experts" (Takahashi, 2014, p. 14).

Based on interviews with three highly respected knowledgeable others in Japan, Takahashi identifies three areas that the knowledgeable others are responsible for:

1. Bringing new knowledge from research and the curriculum
2. Showing the connection between the theory and the practice
3. Helping others learn how to reflect on teaching and learning

(Takahashi, 2014, p. 10).

He also stresses that "simply lecturing about specialized knowledge is not enough", as teachers learn best when theoretical knowledge is tied to "concrete examples from the lesson that they have observed" (p. 11).

A distinctive aspect of the Japanese knowledgeable other's role, as Takahashi (2014) notes, is to support the quality of the post-lesson discussion. One responsibility is therefore "to help the school conduct effective post-lesson discussions" by raising issues "that were not addressed" by the teachers (p. 12).

Takahashi (2014) notes that outside Japan, it is difficult to develop knowledgeable others' expertise because they lack the long-term participation that Japanese experts rely on; as he writes, "there are not enough opportunities to learn how to provide effective final comments" (p. 14). In line with Lewis's observation that "lesson study is easy to do but difficult to do effectively" (as cited in Takahashi, 2014, p. 15), Takahashi (2014) argues that more structured support, such as sharing lesson plans in advance and engaging in meta-level discussions, may be necessary in contexts where lesson study is less established (p. 14).

Alongside the role of the knowledgeable other, the facilitator focuses on guiding the collaborative process itself. As Fernandez and Yoshida (2004) describe, the facilitator leads the post-lesson discussion by opening and structuring the conversation; for example, “Mr. Mizuno [facilitator for the discussion] began by inviting everyone to ask questions or simply voice their opinions” (p. 158) and suggests how the group of teachers might begin analyzing segments of the lesson. According to Winsløw et al. (2018), “the facilitator (a researcher) was present at all times and guided the teachers using questions and suggestions to notice and analyze didactical mechanisms” (p. 132), participating in planning, observation, and post-lesson discussion. Hart and Carriere (2011) further emphasize that facilitators help maintain a reflective space and reduce teacher anxiety, reminding teachers that the discussion is not a review of the teacher’s individual performance but “an assessment of how well the lesson exposed students’ thinking and how well it reached the objectives” (p. 32) and “summarize the discussion, noting themes and issues that emerged” (p. 32). In the Japanese context, facilitators are typically researchers or experienced teachers who ensure coherence between lesson design, observation, and reflections, maintain focus on student learning, and guide attention toward didactical mechanisms rather than teacher performance.

This link between practice and research characterizes the Japanese model of lesson study: “Lesson study has contributed to the culture of dialogue and collaboration between researchers and teachers” (Kusanagi, 2022, p. 170).

Another important aspect is *open-house* events (*kokai jugyō*), where schools open their research lesson study to teachers nationwide (Fernandez & Yoshida, 2004). These events serve as arenas for knowledge sharing, allowing circulation of didactic praxeologies as didactic techniques and technologies are made public to other teachers (Fernandez & Yoshida, 2004; Xu & Pedder, 2015).

From an institutional perspective, lesson study operates as a system in which local experiences are collected, processed, and distributed through a network of schools, researchers, and authorities, connecting local, municipal, and national levels. The most recognized schools can attract hundreds or even thousands of visitors, demonstrating how lesson study in Japan is embedded in a national infrastructure for collective learning and knowledge exchange (Fernandez & Yoshida, 2004; Xu & Pedder, 2015).

Lesson study in Japan can therefore be understood as more than a method – it is a paradidactic infrastructure, in which teachers collaboratively work within the school as an organized professional community. The practice is grounded in professional accountability, where teachers view their work as both an individual and collective contribution to the improvement of teaching and the school itself. As Cheng (2019) notes again: “Just taking the ‘PDCA [plan-do-check-act] Cycles’ of Lesson Study without including teacher professional accountability, the soul of Lesson Study, is doomed to fail, even if we institutionalize it in our schools or our country systems” (Cheng, 2019, p. 61).

This professionalism rests on the idea of *kaizen* – continuous improvement through plan-do-check-act (PDCA) cycles – which in Japan is understood as both a technical method and a professional commitment to continuous improvements among teachers (Cheng, 2019, p. 61). The PDCA cycles can be seen as iterative processes through which teachers continuously develop their praxeologies: adjusting techniques, reflecting on technologies and theories, and integrating experience into the school's collective knowledge base (Cheng, 2019).

In sum, lesson study operates as a paradidactic infrastructure, where schools and broader institutional networks create conditions for collective reflection on teaching techniques and technologies, accumulation of didactic knowledge, and sharing of praxeologies across schools (Kusanagi, 2022; Baba et al., 2018).

Altogether, this shows that lesson study in Japan is part of a larger institutional system consisting of concrete *konaikenshu*-goals, a shared professional language, lifelong professional development, and institutional support from schools, regions, and the government. All of which constitute the paradidactic conditions that enable lesson study to improve teaching and student learning.

4.2 Lesson study beyond Japan

Lesson study has, since the late 1990s, spread beyond Japan and gained international attention as an approach for teacher professional development. While lesson study is a well-established part of teachers' professional lives in Japan, it became a subject of international interest after the publication of *The Teaching Gap* by Stigler and Hiebert (1999).

The book sparked curiosity in the West, particularly in the United States, where researchers sought explanations for Japan's high results in international assessments such as TIMSS and PISA (Dudley, 2014). Within a few decades, lesson study spread to large parts of the world, including North America, Europe, Africa, the Middle East, and Asia (Xu & Pedder, 2015)

One reason for this dissemination is dissatisfaction and disappointment with traditional forms of teacher professional development, such as short-term workshops, which often have a limited impact on teachers' classroom practice (Xu & Pedder, 2015, p. 30)

Lesson study, by contrast, has been seen as a practice-oriented, collaborative, and reflective approach in which teachers develop their teaching through joint planning, observation, and reflection of an actual lesson. Its simplicity allows the model to be adapted to local contexts while maintaining a focus on student learning (Dudley, 2014, p.5).

Several researchers (Kamina and Tinto, as cited in Hart et al., 2011) emphasize, however, that lesson study is not a "one size fits all" model (Hart et al., 2011). Its implementation depends on cultural and institutional contexts, and local adaptations are inevitable (Winsløw et al., 2018).

Inspired by Japanese lesson study, the literature shows various adaptations of lesson study outside Japan. Some misunderstandings have also emerged during the process of introducing lesson study internationally. Kitada (2018) points out that, outside Japan, the focus is often on developing teaching materials and techniques. While this is, of course, necessary for teachers, if it becomes the main purpose of conducting lesson study, it risks diverting attention from its primary aim: students' learning (p. 42).

Seleznyov (2018) reinforces this observation, finding that many international implementations of lesson study omit critical components that define it as a research-based process. For example, 60% of the studies she had reviewed focused on polishing a single "perfect" lesson rather than engaging in repeated research cycles (p. 221), and 61% dit not include any mechanism to mobilize knowledge between lesson study groups (p. 221). This suggests that outside Japan, the lesson study is often simplified and its full research-oriented potential is not realized.

In adapting lesson study beyond Japan, the literature also highlights specific difficulties or constraints that limit its implementation and sustainability. Xu and Pedder (2015), drawing on other researchers' work, notes: "The most frequently mentioned constraints are the lack of time for teachers to engage in LS activities …the extra stress or demand put on teachers to interrogate and refine their practice … and lack of strong leadership support to create favourable conditions for teachers to implement and sustain LS practice" (p. 44).

4.2.1 Implementation of lesson study outside Japan: Difficulties and Constraints

Implementing lesson study outside Japan is associated with a series of structural, cultural, and institutional challenges. Xu and Pedder (2015) emphasize that to sustain lesson study practice in schools and classrooms, well-developed systems of leadership and organizational support are required and that the absence of such systems often hinders its sustainability abroad (p. 47). Murata (2011) states that "modifications are expected and essential" (p. 10) in other cultural and structural contexts when adopting and implementing Japanese lesson study. She identifies four recurring issues in the international implementation of lesson study: the cost of implementation, sustainability, insufficient teacher content knowledge, and the connection to student learning (p. 8). In many countries, releasing teachers for observation requires hiring substitutes, which increases costs, while finding funding and time for external support and grants adds further pressure (Murata, 2011). Furthermore, by contrast to Japan, where lesson study is a continuous, embedded practice, professional development in many Western contexts tends to be discrete and separate programs, which undermines the long-term culture of collective inquiry that lesson study depends on (Murata, 2011). In addition, in Japanese lesson study, "textbooks provide substantial support for study of curriculum content" (Lewis et al., 2016, p. 576), forming a central part of the planning phase; outside Japan, such structured support is often absent. This absence is evident, for example, in the work of US teachers with lesson study, where a common misconception is that it "entails writing a lesson from scratch, rather than careful study of the existing curriculum and of how to best teach students (Lewis, 2016, p. 579).

Another major difficulty concerns the lack of experienced lesson study experts or *knowledgeable others* who play a crucial role in Japan. Seleznyov (2018) finds that 55% of international studies that she reviewed did not involve a knowledgeable other (p. 221), highlighting that the absence of external expertise is a recurring challenge in implementing Japanese lesson study beyond Japan. LO (2019) notes that outside Japan, such experienced lesson study experts are uncommon because the professional environment needed to develop lesson study experts does not yet exist (p. 805).

Hart et al. (2011) further emphasize that becoming a knowledgeable other requires adopting a learning stance, in which leaders and experienced facilitators see themselves as co-learners with the participating teachers, rather than as external trainers imparting knowledge (p. 289). This shift can be difficult for educators used to directive professional development, which partly explains why lesson study facilitation has been challenging to establish outside Japan.

Cheng (2019) connects this challenge to broader cultural differences: Japan's high-trust society fosters knowledge sharing and enables the Japanese to engage in professional dialogue and collective reflection, while lower-trust environments struggle to replicate these dynamics.

Cheng (2019) also argues that successfully institutionalizing lesson study requires cultural change within the education system involving shared vision and consensus among school leaders, teachers, and parents. Lewis (2006) further stresses that lesson study should not be treated as a "recipe" but as a flexible learning system that must adapt to local educational contexts (p. 12). When lesson study practices do not align with local school traditions or teacher cultures, implementation may be limited in depth or occur only temporarily. Taylor et al. (2005) (as cited in Kusanagi, 2022, p. 56) report that U.S. teachers felt empowered by lesson study but also frustrated by external constraints, while Dudley (2013) (as cited in Kusanagi, 2022, p. 56) notes that U.K. school leaders were discouraged by timetable disruptions, staff coverage issues, and budget limitations.

Moreover, Hart et al. (2011) note that when teachers are required to participate in lesson study without understanding its purpose or value, their lack of buy-in can undermine the process and spread negative attitudes within the community (p. 289). Similarly, Kusanagi (2022) finds that even when lesson study is introduced successfully, if it clashes with existing school routines or teacher norms (i.e., the practices that the teachers typically follow), it rarely becomes a sustainable practice (p. 57).

Overall, these findings reveal that the effectiveness of lesson study outside Japan is often constrained by limited time, financial resources, cultural incompatibility, and insufficient institutional support.

4.2.2 Adaptation of Lesson Study

Experiences from various countries show that implementing lesson study outside Japan inevitably involves contextual adaptation. Winsløw et al. (2018) argue that there is no universal lesson study model; rather, lesson study practices are shaped by cultural and institutional conventions. According to Lewis (2004), successful adaptation requires a deep understanding of what makes lesson study useful to Japanese teachers and how it can be adapted and meaningfully transferred to new contexts (p. 134).

Clivaz and Takahashi (2018) argue that a major reason for these misunderstandings is the lack of a comprehensive understanding of Japanese lesson study outside Japan (p. 155). International adaptations often focus on its visible components, the research lesson and post-lesson discussion, while overlooking the less visible but crucial stages, such as the detailed design of the lesson, the lesson plan, and the rationale behind it (Fujii, 2018). This partial understanding tends to produce simplified or procedural versions of lesson study that differ significantly from the Japanese approach. (Clivaz & Takahashi, 2018, p. 155)

Similarly, da Ponte et al. (2018) maintain that transforming lesson study is both inevitable and necessary when introduced to different educational cultures: while certain elements of the original practice may be lost, others may emerge that make lesson study more suited to local conditions. Seleznyov (2018) supports this, concluding from a review of 97 international studies that many implementations lack critical components such as the identification of a research theme, study of materials during planning, repeated cycles of research, and mobilization of knowledge between lesson study groups. She emphasized that outside Japan, there is no internationally shared understanding of Japanese lesson study, which often leads to partial or simplified practices (p. 223). Concrete examples illustrate these adaptations. In Portugal, lesson study was modified so that during teacher education, the research lesson was taught not by the prospective teacher but by another teacher involved in the program (da Ponte et al., 2018). In Ireland, Ní Shúilleabháin (2018) found that lesson study effectively supported mathematics curriculum reform by allowing teachers to plan, observe, and reflect collaboratively on new instructional approaches.

In the case studies from Portugal, Ireland, Denmark, and the USA, the lesson study processes were often university-led, with faculty members acting simultaneously as facilitators and knowledgeable others, a dual role quite different from Japan, where experienced teachers usually assume these positions (Clivaz & Takahashi, 2018, p. 157)

One of the most debated adaptations concerns the practice of re-teaching. Fujii (2018) clarifies that "the most serious misinterpretation of lesson study is the question, ""Should a research lesson always be re-taught?"" (p. 17). In Japan, trial lessons are rarely repeated; they occur only in situations, such as before large, nation-wide research lesson events, to fine-tune lesson plans based on students' initial responses. As Fujii (2018) explains: "While it is true that, sometimes, Japanese teachers will perform trial lessons, this is different from simply re-teaching the whole lesson … The teacher will use this trial lesson to fine-tune their plan in light of students' actual responses" (p. 17).

This misunderstanding can be traced back to how Stigler and Hiebert (1999) presented the lesson study process in The Teaching Gap. They described the final step of Lesson Study as "teaching the revised lesson," which has sometimes been interpreted as requiring that the same research lesson be re-taught after revision (Fujii, 2018). These trial lessons are not repetitions but opportunities for deeper reflection, whereas in some Western adaptations, the "re-teaching" has mistakenly become a routine step, shifting focus away from analyzing authentic student learning in the first research lesson (Fujii, 2018; Clivaz & Takahashi, 2018).

Another critical aspect often overlooked in international adaptations of lesson study is *kyozaikenkyu*, the in-depth study of teaching materials and content. Seleznyov (2018) highlights that 63% of international implementations of lesson study did not include kyozaikenkyu (p. 221). Fujii (2018) stresses that this process is fundamental to the Japanese lesson study cycle, even if it remains largely invisible to outsiders. Fujii (2018) emphasizes that teachers outside of Japan who wish to implement lesson study must "both understand and successfully implement kyozaikenkyu" (p. 16). This involves engaging with the mathematical and didactical aspects of the content, asking questions such as "What does this idea really mean? How does this idea relate to other ideas? What is/are the reason(s) for teaching this idea at this particular point in the curriculum? What ideas do students already understand that can be used as a starting point for this new idea?" (Fujii, 2018, p. 16). Without this process, lesson planning risks becoming superficial, reducing lesson study to a procedural exercise rather than a vehicle for professional inquiry.

In Japan, the lesson plan is interpreted as a learning proposal where deviations between the lesson plan and the lessons are "never thought to be wrong" (Fujii, 2014, p. 8). In countries outside Japan, however, the lesson plan is often treated as a script, so that any deviation may be viewed negatively.

As Winsløw et al. (2018), understanding lesson study through an institutional lens — such as the Anthropological Theory of the Didactic (ATD) — can clarify how lesson study interacts with broader educational structures, helping researchers distinguish authentic lesson study from superficial "lesson study-like" practices.

Taken together, these international experiences suggest that lesson study has the potential to support professional development and curriculum reform — but only when it is adapted thoughtfully to local conditions, supported institutionally, and rooted in the cultural and social realities of the context in which it operates.

4.3 Lesson study across institutions

A review of the identified literature shows that only a small number of studies explicitly address lesson study across institutions or in relation to transitions between them. Although lesson study is widely used for teacher professional development, most research focuses on collaboration within a single school. The following studies illustrate the few cases touching on collaboration across institutions.

Lundbäck and Egerhag (2020) provide an example of an attempt to use lesson study across two different learning contexts within the same institution: elementary school and school-age educare for grades 1-3. Their aim was "to capture the collaboration between teachers in these two contexts" (p. 291) and to explore how experiences might "change teaching in the two situations so that school-age educare becomes a greater asset for the school" (p. 291).

Despite this, collaboration remained limited: "the teachers in this study planned common content and discussed how the topic could be presented to students in both contexts, they did not work together in the actual lessons" (p. 292). The teachers nonetheless emphasized that collaboration "should be permanent, as they believe that teachers can learn from each other through collaboration" (p. 298). However, the study only mentions collaborative lesson study as a theoretical possibility rather than documenting an actual practice of planning and observing lessons together, across contexts.

Mynott and O'Reilly (2022) examine collaboration within lesson study by reviewing existing literature on small-scale studies. Their aim was to identify what could be learned about collaboration in lesson study from published research. They note that "the most surprising finding of this research was the very small number of publications that provide detailed exemplification of collaboration in LS" (p. 184). Collaboration is often mentioned but poorly defined or explained (p. 184). They conclude that "no single publication provides an effective overview of the layers of collaboration that has occurred" (p. 184). Neither the paper nor the meta review takes the collaborating partners' institutional affiliation into account, so there is no evidence of lesson study across institutions in the literature reviewed.

Joubert et al. (2020) demonstrate lesson study across geographical distance within a blended professional development program, by combining face-to-face and online aspects, for teachers from southern Africa. Their research aimed to identify "which aspects are necessary to support these isolated teachers through all the phases of LS in a blended approach" (p. 922). They found that online and blended environments could "bring isolated teachers together through online interaction and 'teamwork'" (p. 922) and that lesson study could function on a relatively large scale "despite distance between participants" (p. 924). This study addresses collaboration across distance but within the same school institution.

Other studies provide knowledge about collaborative professional development beyond lesson study and with other purposes than to address transitions. For example, Goodchild (2014) focuses on the establishment of communities of inquiry involving teachers and didacticians across multiple schools, but this work was not framed as lesson study and did not focus on relations between neighboring institutions or transitions.

Similarly, McGraw et al. (2007) analyzed professional discussions among teacher groups, including "in-service teachers, pre-service teachers, mathematicians, and mathematics teacher educators" (p. 97), as part of a larger project with the long-term aim of establishing lesson study groups. The reported research does not examine lesson study nor focus on transitions. Rather, it explores how group compositions shaped participants' noticing during discussions of a multimedia case of classroom practice, "what it was that the group members did notice, which aspects of what they noticed then became the focus of dialogic interaction" (p. 98).

Taken together, these studies show that while lesson study is linked to collaboration and teacher professional development, it is rarely used to investigate or support collaboration between different institutions and never (in these references) to explore students' transitions between them. So, at least in the three journals considered here, there are no reports on research into lesson study as a means to address transition problems between institutions.

5 Literature review – transitions and school algebra

A recent bibliometric review by García Fajardo et al. (2025) shows that the number of publications on the transition from arithmetic to algebra and the challenges involved in this transition has grown since the early 2000s, particularly on early algebraic thinking, understanding variables and equality, and difficulties in the arithmetic-algebra transition.

Research on this topic can be traced back to at least the 1980s, when Usiskin (1988) introduced different conceptions of school algebra (which will be elaborated on later). Soon after, Filloy and Rojano (1989) identified a "breaking point" in equation solving, where students must abandon arithmetic strategies and adopt a more algebraic approach (García Fajardo et al., 2025, p. 9).

In the early 1990s, Kieran (1992) integrated previous research into a framework describing how algebraic thinking develops from early grades (García Fajardo et al., 2025, p. 9). Herscovics and Linchevski (1994) further elaborated on the cognitive gap between arithmetic and algebra. From the 2000s, research moved toward more systematic models (García Fajardo et al., 2025). Kieran (2004) introduced algebraic thinking through three types of activities and Kaput (2008) defined algebraic thinking as the ability to formulate and express generalizations in symbolic systems.

These contributions form the basis of the following literature review of arithmetic and algebraic thinking and the transitions between them.

5.1 Relations between arithmetical and algebraic thinking

Usiskin (1988) identifies four conceptions of school algebra, each defined by a different use of variables. In *generalized arithmetic,* variables act as generalizers that represent ranges of values in order to express general rules and properties such as the commutative or distributive property; for example, $a + b = b + a$ uses variables a and b to express a general property of addition applicable to all numbers. In *algebra as problem-solving procedures,* algebra is viewed as a system of rules and procedures for solving and simplifying problems, with variables functioning as unknowns or placeholders for specific, initially unknown numbers; for instance in $5x + 3 = 40$, the variable x represents an unknown value to be determined. In *studying relationships among quantities,* variables represent arguments, parameters, or varying quantities, introducing distinctions such as independent and dependent variables; for example, in $y = 3x + 5$, x is the independent variable, and y depends on x). Finally, in *algebra as the study of structures,* variables are abstract symbols within axiomatic systems such as groups or rings, where the focus lies on operations and structural properties rather than numerical interpretation; for example, in the group-theoretic expression $x + y = y + x$, the variables x and y represents elements of a group, illustrating the structural property of commutativity.

Kieran (1992) distinguishes between a procedural and a structural perspective in algebra, and both are important in the teaching and learning of algebra (p. 414). Procedural thinking refers to carrying out arithmetic operations by substituting numbers into algebraic expressions and computing with numbers, for example, substituting $x = 4$ and $y = 5$ into the expression $3x + y$ to obtain 17 (p. 392). By contrast, structural thinking of algebra refers to operating on algebraic expressions themselves rather than on numbers. For example, the expression $3x + y + 8x$ can be simplified to $11x + y$. Here, the object being manipulated is the algebraic expressions, and the results remain algebraic rather than numerical (p. 392).

In 2004, Kieran proposed an activity-based model dividing early algebraic thinking into *generational*, *transformational*, and *global meta-level* activitie*s*. Generational activities involve forming algebraic expressions and representing problems as equations (Kieran, 2004, p. 142). Transformational activities involve typical algebraic manipulation such as collecting like terms, simplifying expressions, or solving equations (Kieran, 2004, p. 142). Global meta-level activities refer to activities where "algebra is used as a tool but which are not exclusive to algebra" (Kieran, 2004, p. 142). These activities include problem solving, modelling, noticing structures, generalizing, justifying, predicting, and analyzing relationships, which can be conducted without using algebra (Kieran, 2004, p. 142). Kieran (2004) defines algebraic thinking in the early grades based on the global meta-level activity of algebra as follows:

> *Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (p. 149).*

Kaput (2008) provides a symbolization perspective on algebraic reasoning, describing it as "complex symbolization processes that serve purposeful generalization and reasoning with generalization" (p. 9). He distinguishes two core aspects: *Core Aspect A,* algebra as systematically symbolizing generalizations of regularities and constraints, and *Core Aspect B,* algebra as syntactically guided reasoning and manipulation of generalizations expressed in conventional symbol systems (p. 11). One might observe potential parallels to Kieran's model: generational activities share features with Core Aspect A, as students construct algebraic objects and express generality, while transformational activities resemble Core Aspect B, focusing on syntactically guided manipulation of symbols. Kaput further emphasizes that students should first explore generalization using their own representations before adopting conventional notation (pp. 11-12).

Kieran (2022) further refines the framing of early algebraic thinking with three dimensions: analytic, structural, and functional thinking, with generalizations as the common thread "that runs through the three dimensions, treating it as separately proved unfeasible" (p. 1134). Analytic thinking, rooted in the generalized-arithmetic perspective, is closely linked to Radford's notion of *analyticity.* According to Radford, analyticity is "central to algebraic thinking – the aspect that distinguishes it from arithmetic thinking" (Kieran, 2022, p. 1134). It involves working with indeterminate quantities (unknowns or variables) "as if they were known" (p. 1134) and carrying out calculations with them as with known numbers, with emphasis on their deductive manipulation (Kieran, 2022, p. 1144, citing Radford, 2014, 2018). From this perspective, analytic thinking does not require explicit algebraic notation; students can reason about indeterminate quantities using natural language, gestures, or unconventional signs, as long as they treat unknowns as if they were known (Kieran, 2022).

Structural thinking, also rooted in the generalized-arithmetic perspective, concerns seeing and expressing relations, structures, and properties within numbers, operations, and expressions (Kieran, 2022, p. 1144). Kieran notes that "relationships and properties are at the heart of structural thinking" (p. 1138), drawing on Kaput's description of the generalized-arithmetic as involving "generalizing arithmetic operations and their properties, and reasoning about more general relationships from the structure of arithmetic" (Kaput, 2008; cited in Kieran, 2022, p. 1138).

Functional thinking positions function as the central mathematical object. Drawing on Kaput (2008), Kieran (2022) explains that functional thinking involves "describing systematic variation of instances across some domain" (p. 1141). Blanton et al. (2011, as cited in Kieran, 2022) add that functional thinking requires "generalizing relationships between co-varying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior" (p. 1141), i.e., identifying the underlying rule or relationship linking the co-varying quantities.

Across all three dimensions, generalization is what "stitches together the three dimensions, albeit with its own particular flavor in each of them" (Kieran, 2022, p. 1134). In analytic thinking, generalization concerns equivalence and the behavior of unknowns; in structural thinking, it concerns general properties and relationships within arithmetic and algebraic structures; and in functional thinking, it concerns general relationships between co-varying quantities.

Functional thinking overlaps with Kieran's earlier-mentioned global meta-level and Kaput's modeling strand, where algebra serves as a tool to "explore its more general form, scope, and deeper relationships – including comparison with other models" (Kaput, 2008, p. 14). Thus, Kieran (2022) extends the 2004 model by combining activity types with explicit analytic, structural, and functional dimensions, with generalization as the guiding line throughout these three dimensions of algebraic thinking.

5.2 The transition from arithmetic to algebra

Filloy and Rojano (1989) describe the transition from arithmetic to algebra, which involves developing new concepts, representations, and ways of operating on symbols (p. 19). This transition does not occur smoothly but involves specific *cut-points,* where arithmetic strategies become insufficient, and students must begin to develop "some elements of an algebraic syntax" (p. 19). These cut-points arise when students face operational insufficiency: arithmetic "undoing" strategies no longer work, and students must instead "operate on what is represented" (p. 20), manipulating symbols rather than concrete numbers.

For example, a secondary school student may solve equations such as $x + 5 = 18$ (on the form $Ax + B = C$) by undoing the operations – subtracting 5. Here, x is not treated as something one operates on. But when the student encounters equations like $7x + 2 = 3x + 6$ (on the form $Ax + B = Cx + D$), the earlier used "undoing" technique becomes insufficient, and students must operate on what is represented (Filloy & Rojano, 1989, p. 20).

A central difficulty lies in students' arithmetic conceptions of equality, in which an equation is interpreted as a sequence of operations leading from left to right, with the left-hand side representing the operations to be carried out and the right-hand side representing the consequence, for example, as the earlier-mentioned equation $x + 5 = 18$. This view is inadequate in algebra, where both sides must be seen as structurally equivalent expressions. Secondary school students also do not spontaneously attempt to operate on unknowns, and when encountering equations such as $Ax + B = Cx$, guess-and-try strategies typically dominate (Filloy & Rojano, 1989).

Herscovics and Linchevski (1994) extend this perspective by identifying a *didactic cut:* a sharp delineation that appears when the unknown occurs on both sides of a first-degree equation. Although many students could solve such equations without any prior instruction, their guess-and-try strategies revealed a cognitive gap characterized by the students' "inability to operate spontaneously with or on the unknown" (p. 63). Their findings further underscore the need to develop students' understanding of the equal sign and to expand the arithmetic notions that are essential for later algebra (p. 75).

The equal sign and students' conception of it are important in the transition from arithmetic to algebra. Students encounter the equal sign early, often as early as pre-school. Kieran (1981) notes that sixth-grade students view it as a "do something signal" (p. 319), primarily as an operator rather than a relational symbol. For many 13-year-olds, "this is a transition period-transition between requiring the answer after the equal sign and accepting the equal sign as a symbol for equivalence" (p. 320), and 12- to 14-year-olds frequently describe it as representing the answer (p. 321). This operational conception of the equal sign can hinder students when they face algebraic equations in which the right-hand side is not simply the answer but an expression equivalent to the left-hand side. As Kieran (1981) emphasizes, "the reason for extending the notion of the equal sign to include multiple operations on both sides was to provide a foundation for the later construction of meaning for non-trivial algebraic equations" (p. 322). "Multiple operations" refers to arithmetic expressions with more than one operation, like $7 \cdot 2 + 3 - 2 = 5 \cdot 2 - 1 + 6$, and "non-trivial" algebraic equations refer to equations where unknowns appear on both sides (p. 321). Developing a relational conception allows students to view both sides as equivalent, enabling them to construct and interpret equations such as $3x + 5 = 2x + 12$ and not only equations as $3x + 5 = 26$ will fit in their existing notions (p. 321). A relational conception of the equal sign is therefore essential in the transition from arithmetic to algebra.

Taken together, these studies show that the transition from arithmetic to algebra, among other things, requires students to move from equations solvable by undoing operations ($Ax \pm B = Cx$) to equations that require operations on unknown ($Ax \pm B = Cx \pm D$). Because of this shift, from an operational to a relational understanding of equality, which typically occurs as students move from primary to lower secondary school, it is often this transition that gives difficulties.

5.3 School algebra within ATD

School algebra has been studied from an ATD perspective for many years and has been the core of the development of ATD since its origin (Chevallard, 1985, as cited in Bosch, 2015; Bosch, 1994, as cited in Bosch, 2015). According to Bolea et al. (2004), students' first encounter with algebra draws on arithmetic, which serves as a reference point for their algebraic work (p. 126) and contributes to the dominant understanding of school algebra as *generalized arithmetic.* In this view, algebra is presented as an "algebraic language", a symbolic system used to express arithmetical rules in general terms, while the operations remain identical to those of numerical calculations. Bolea et al. (2004, p. 126) identify four types of tasks that reflect this conception:

1. Writing numerical expressions with symbols that describe and generalize arithmetical calculation techniques.
2. Manipulating algebraic expressions by simplifying or transforming them.
3. Establishing and manipulating algebraic expressions where the letters represent unknown numbers.
4. Solving word problems with equations through a translation of the verbal problem by assigning a name to the unknown quantities and numerical values.

An alternative perspective identified by Bolea et al. (2004) considers algebra as a *modelling tool* within an algebraization process (to be elaborated further below). Here, algebra is not treated as content in itself but as a general modelling tool for all mathematical organizations in school (including arithmetic, geometry, or statistics) (Bolea et al., 2004; Bosch, 2015; Ruiz-Munzón et al., 2013). Tasks in this approach typically involve five characteristics (Bolea et al., 2004, p. 127):

1. Algebra is used to model and solve problems in other mathematical domains, such as arithmetic or geometry, that would otherwise be difficult to handle within those domains alone.
2. Algebraic modelling clarifies the scope, reliability, and justification of the mathematical activity by describing, generalizing, and justifying problem-solving processes and unifying techniques that initially appear unrelated.
3. Modelling expands the original system by generating new types of problems, techniques, and new interpretations.
4. Letters represent quantities rather than merely unknown numbers, and algebraic manipulation does not rely on a strict distinction between knowns and unknowns.
5. This process includes tasks in which students explore relations between different magnitudes and evolve towards functional modelling.

5.3.1 Algebra as a process of algebraization

According to Ruiz-Munzón et al. (2013) a reference epistemological model (in the sense of Chapter 2.3) for school algebra is proposed by ATD that begins with arithmetical praxeologies and their techniques centered around calculation programmes (CPs), defined as "a sequence of arithmetic operations applied to an initial set of numbers or quantities that can be effectuated "step by" and provides a final number of quantity as a result" (p. 4). Within this model, the algebraization process represents the transformation of arithmetic praxeologies into algebraic ones through three stages (Ruiz-Munzón et al., 2013, p. 5; Bosch, 2015, pp. 62-63):

- Stage 1: A CP must be considered not only as a process but as a whole. This does not necessarily involve the use of letters but requires making the structure of the CP explicit, including the hierarchy of operations. The focus is on creating and simplifying algebraic expressions by "simplifying" and "transposing" equivalent terms.
- Stage 2: Algebraization proceeds when identities between CPs need to be manipulated. This includes considering equations and the techniques required to solve them, including equations with both an unknown and a parameter, where the solutions are expressed as relationships between the involved variables. When one numeric argument has a concrete value, the problem reduces to solving a one-variable equation.
- Stage 3: Algebraization reaches its final stage when the CP can have an arbitrary number of variables and the distinction between unknowns and parameters is eliminated.

These three stages not only provide a description of how algebra emerges from arithmetic praxeologies but also an analytical tool for examining which forms of algebra are taught and learned, as well as what is omitted or could be developed under specific conditions.

The reference epistemological model also serves as a framework for analyzing algebraic praxeologies and their ecological effect, that is, the constraints and conditions they create for other mathematical content areas (Ruiz-Munzón et al., 2013, p. 6).

6 The context of the study

The context of this Ph.D. project is twofold. First, the intervention takes place in a Danish lower secondary school (grades 7-9) and involves teachers from this institution, while teachers from a Danish upper secondary school are also involved in this intervention. Given this context, it is important to examine the educational backgrounds of these teachers, as this shapes both their pedagogical, didactic, and subject-specific knowledge. The secondary focus of the Ph.D. project concerns how arithmetic and algebra appear in the Danish context. These two aspects will be elaborated in detail in the following Chapters, as space constraints prevented a thorough discussion in the papers.

6.1 Teachers' different educational backgrounds

Primary and lower secondary school teachers

To qualify as a primary and lower secondary school teacher in Denmark, students complete a four-year professional bachelor's degree at a University College, which integrates theoretical coursework with school-based teaching practice. Teachers are educated to teach all grade levels in the 10-year Danish public school (grades 0-9).

The program comprises 240 ECTS, including

- 70 ECTS of core pedagogical studies (covers pedagogy and general didactics, educational psychology and inclusion, citizenship and humanities, Danish as a second language, and a small elective)
- 120 ECTS of teaching subject, typically distributed across three subjects (mathematics, together with Danish, counts for 50 ECTS)
- 40 ECTS of practicum
- 10-ECTS bachelor project.

This structure shows that a substantial part of the program focuses on pedagogy, general didactics, and the understanding of pupils, learning processes, and school culture (Uddannelses- og Forskningsstyrelsen, 2024).

In comparison, the mathematics component (50 ECTS), representing approximately one-fifth of the total program, focuses on:

- The mathematical content of the curriculum
- General mathematics didactics (teaching methods, learning materials, assessment, and differentiation)
- Subject-specific didactics (such as progression in teaching and common subject-related misconceptions among primary- and lower secondary school students)
- Formal requirements of the national curriculum and mandatory tests

(Uddannelses- og Forskningsstyrelsen, 2024).

Overall, the Danish teacher education program places substantial emphasis on pedagogy and didactics, while academic depth in individual subjects is less prominent.

Upper secondary school teachers

Candidates seeking a permanent position in Danish upper secondary schools (gymnasium) in Denmark must hold a master's degree, usually in two subjects (major and minor). They also have to complete the one-year (60 ECTS) induction programme ("Pædagogikum"), which includes elements of practicum, general pedagogy, and subject matter didactics.

There are two main routes to becoming a mathematics teacher at the Danish upper secondary school

1. Master's degree in mathematics with a minor in another subject
2. Minor in mathematics combined with a Master's in another subject.

To satisfy the national minimum requirements for upper secondary mathematics teachers, candidates must complete 120 ECTS in mathematics, distributed as follows:

- Core content (minimum 60 ECTS): differential and integral calculus (including differential equations), analysis, geometry, linear algebra, algebra, probability and statistics, and discrete mathematics
- Depth content (up to 30 ECTS): advanced topics that extend the core content, including mathematical modelling
- Breadth content (ca. 20 ECTS): history of mathematics, programming related to core topics, and mathematical applications in other subjects
- Subject didactics and philosophy of mathematics (ca. 10 ECTS).

(Uddannelses- og Forskningsministeriet, 2018).

Furthermore, the bachelor's project (15 ECTS) and master's project (30 ECTS) can be written in Didactics of Mathematics if students have completed the corresponding courses.

Compared with Danish primary and lower secondary teacher education, upper secondary teacher education places much greater emphasis on subject matter knowledge. In a mathematics major at the University of Copenhagen, candidates typically complete around 165 ECTS in mathematics courses across the bachelor's and master's programs. By contrast, a primary and lower secondary school teacher specializing in mathematics completes 50 ECTS in mathematics out of a total of 240 ECTS, with the majority of the program devoted to pedagogy and didactics.

This contrast reflects the differing emphases of the two pathways: the upper secondary education prioritizes in-depth subject knowledge, while the primary and lower secondary education focus more on pedagogical and didactic knowledge.

6.2 Arithmetic and algebra in the Danish context

The arithmetic and algebraic knowledge of Danish students has for many years been the focus of investigations, development projects, and research initiatives. National evaluations, such as national tests and examinations, as well as international assessments including TIMMS and PISA, indicate that a substantial part of Danish students experience particular difficulties in mathematics, particularly in arithmetic. TIMSS 2019 documents "a statistically significant decline in Danish fourth graders' mathematics achievement" (Kjeldsen et al., 2019, p. 17), including "a drop of 23 score points in the domain *Tal* [arithmetic] from 2015" (Kjeldsen et al., 2019, p. 419). Likewise, the PISA 2022 report a growing share of low-performing students in mathematics, rising from "14.6% in 2018 to 20.4% in 2022" (Jóelsdottir & Østergaard, 2023, p. 7). These documented difficulties in mathematics, particularly arithmetic, appear early in school and persist, often intensified, at later educational stages.

An expert group under the Danish Ministry of Children and Education has identified the area of arithmetic and algebra [*Tal and algebra in Danish*] as a central and urgent issue for mathematics education in both lower secondary school and upper secondary school, as described in the report *Fælles udvikling af matematik [Joint development of mathematics]* (Skott et al., 2022).

The report further emphasizes that transition problems between these school levels are closely related to students' difficulties in arithmetic and algebra: "Several of the transition problems should be seen in connection with challenge 2 concerning numbers and algebra. A larger proportion of students entering upper secondary school today thus do not have sufficient prerequisites in numbers and algebra" (Skott et al., 2022, p. 33, Author's translation).

The report also emphasizes, in particular, that insufficient progression in students' learning of arithmetic and algebra leads to barriers for their further mathematical development (NCUM, 2025).

The transition from lower secondary to upper secondary school constitutes a particularly critical point. In upper secondary school, algebra assumes a much more prominent role than in lower secondary school, and limited knowledge of algebra can hinder students' continued learning in other domains of mathematics. In vocational school as well, working with formulas, measurement units, and applied calculations presupposes a solid understanding of arithmetic and algebra. This underscores the importance of coherence and progression in students' learning across educational levels (NCUM, 2025).

The project *Matematikbroen* described in Jessen et al. (2016) addresses this issue by investigating how collaboration and professional coordination between Danish lower secondary and upper secondary mathematics teachers can help alleviate transition-related difficulties. The project aimed to develop teaching materials and interventions for the final years of lower secondary school. It targeted areas that often create difficulties at the beginning of upper secondary school, particularly elementary algebra (symbol manipulation, equation solving, etc.), as well as basic mathematical modelling and the application of mathematics in context. The results indicate that systematic collaboration can help mitigate parts of the transition problem and support students' mathematical preparedness (Jessen et al., 2016).

Research-based analyses of Danish students' concrete mathematical knowledge point to misconceptions and a lack of knowledge within arithmetic and algebra.

The purpose of Poulsen's (2015) master's thesis was to identify the algebraic techniques involved in the transition from lower secondary school to upper secondary school and to examine how well students master these techniques upon entering upper secondary school. The study mapped which techniques overlap between the two levels, while test results showed that many students still struggle with basic algebra. One of the tests revealed several misconceptions, including difficulties with notation and the use of symbols. The thesis also suggests that the increasing use of IT tools may reduce students' practice with algebraic techniques.

Cosan's (2021) master's thesis examined Danish fifth- and seventh-grade students' arithmetic and algebraic knowledge using a diagnostic test and identified, among other things, a prominent misconception related to fractions, where students treated fractions as whole numbers, for example, when comparing the sizes of fractions. In one of the tasks in the test, students were asked to place $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ on a number line. The most common answer was to place $\frac{1}{2}$ first, followed by $\frac{1}{3}$, and then $\frac{1}{4}$. Cosan (2021) points out that this may be because students likely view fractions as two separate parts and as whole numbers. They may reason that since 2 comes before 3 as whole numbers, the same order should apply to the fractions (p. 74).

Furthermore, students show widespread misconceptions regarding addition, subtraction, and multiplication of fractions, with some students adding or subtracting numerators and denominators separately, even when the denominators differ. For instance, only $\frac{1}{3}$ of seventh-grade students could give a correct answer for the calculation $\frac{7}{8} - \frac{3}{4}$ and a common wrong answer among them was $\frac{4}{4}$ (Cosan, 2021, p. 88).

Cosan (2021) also found that fifth-grade students have difficulties in solving equations as the one $x - 36 = 48$. Only about 20-35% of the students gave a correct answer (p. 81). Also, when students have to describe what the equal sign is, they show an operational approach to the equal sign by saying "what something gives" (p. 82).

Tonnesen (2025) further contributes to this line of research by examining the transition from arithmetic to algebra in Danish public schools. Her dissertation investigates whether a research-based textbook resource, specifically a translated chapter from a Japanese mathematics textbook, can support Danish teachers in teaching school algebra and in strengthening students' progression from arithmetic to algebra. Tonnesen (2025) finds that teachers face both expected challenges and unexpected potentials in the materials. They were able to use the problem in the introduction of the book to introduce algebra as a modelling tool and work effectively with notational conventions. The concrete tasks in the material are manageable, but the more theoretical goals are harder to achieve.

Many of these difficulties persist in upper secondary school. Data from the Mathematics B-level exam (taken by students completing their second year of the three-year upper secondary program) in summer 2019 showed that the passing threshold had to be lowered to 20.5% correct answers, illustrating ongoing challenges in students' mastery of algebraic technique (Grønbæk et al., 2019). They also point out that only 24% of a sample of 125 students solved the following task correctly: "Reduce the expression $(a + b)^2 - b \cdot (2a + b)$" (p. 81).

Grønbæk et al. (2019) further note that data from the final exam in lower secondary school show that around 60% of students master only the most elementary operations with fractions.

Taken together, these findings provide strong arguments for maintaining a sustained research and policy focus on arithmetic and algebra in Denmark. The challenges appear systemic, persistent, and deeply connected to students' opportunities to develop robust mathematical competencies throughout their educational trajectories.

7 Research questions

As outlined in the literature review, research on school algebra has highlighted the difficulties in the transition from arithmetic to algebra for students. These findings align with the Danish context, where national and international assessments also highlight students' difficulties in algebra and the transition from lower secondary to upper secondary school in algebra.

As pointed out in the introduction (Chapter 1), Gueudet (2016) notes that initiatives aimed at supporting students' transition often focus on strengthening relationships between teachers across the different institutions. This observation motivates this PhD project's interest in teacher collaboration as a potential means to address the transition problems. While lesson study has a long and well-established tradition in Japan and has been adapted internationally, there are, as discussed in Chapter 4.3, no studies that have examined lesson study conducted across two neighboring institutions with a specific focus on a subject-specific transition problem. This gap in the literature frames the current project, which seeks to both identify the transition problem in algebra in the Danish context and explore how a collaborative approach can support teachers in addressing it.

As stated in the introduction, the overall objectives of the project are twofold: (1) to develop a general methodology for identifying transition problems and to investigate the transition problem in algebra from Danish lower secondary to upper secondary schools, and (2) to experiment with a bi-institutional lesson study format and examine how such collaboration may help address the transition problem.

The research questions presented here are largely inspired by those stated in the original PhD application to the Council of Education Research, with a few modifications that will be explained below. From the outset, I have been interested in the transition problem in algebra, which, as highlighted in the Context (Chapter 6.2), has been a longstanding concern in the Danish school system. Initially, I defined this problem as a "gap" between lower and upper secondary school algebra. However, during my PhD, the theoretical framework I work within made it possible to specify this gap more precisely in terms of praxeological differences, that is, differences in the types of tasks, techniques, technologies, and theoretical knowledge emphasized as students move from one institution to the next.

This leads to the first question:

RQ1: What are the praxeological differences between the algebraic praxeologies currently taught and learned in Danish lower secondary school and those expected at the entrance of Danish upper secondary school in relation to the algebra transition problem? What methods can be used to identify such differences?

As mentioned in the introduction, a second focus of this PhD has been to experiment with a lesson study format in which lower and upper secondary teachers collaborate to address the transition problem and support students in the transition. Inspired by my original research question on how lesson study could be used to develop new didactic praxeologies for middle school algebra, it became necessary first to explore how such collaboration could be established, given the different institutional and educational backgrounds of the teachers, as described in the Context (Chapter 6). This inquiry leads to the research question:

RQ2: What conditions and constraints appear to facilitate the establishment and process of bi-institutional lesson study?

Finally, building on the original research question in the application, the third focus of this PhD is to examine how teachers, through such collaboration, can develop new didactic and mathematical praxeologies to address the transition problem:

RQ3: In view of reducing specific praxeological differences in algebra (as identified in RQ1), how does bi-institutional lesson study function as a paradidactic infrastructure to develop relevant didactic praxis and theoretical knowledge related to these praxeological differences?

Together, these research questions form a clear progression: RQ1 identifies the transition problem in algebra, RQ2 examines how a collaborative paradidactic infrastructure can be established to address this problem, and RQ3 examines how such an infrastructure can support teachers' professional development and their development of didactic techniques related to the praxeological differences.

A more thorough examination of how the papers address the research questions is presented in Chapter 9; in short, the correspondence is one-to-one. RQ1 aligns with the focus of Paper 1, as both examine how praxeological differences can be identified and, subsequently, what characterizes the differences between Danish lower and upper secondary schools. RQ2, which concerns the establishment of a bi-institutional lesson study, directly aligns with the research questions and empirical work in Paper 2. RQ3 corresponds primarily to the research questions and work in Papers 3 and 4, focusing on teachers' development of didactic praxis and theoretical knowledge related to praxeological differences in algebra when collaborating within a bi-institutional lesson study.

8 Methodology

8.1 Methodology for RQ1

Before presenting the methodology for identifying the praxeological differences between Danish lower and upper secondary schools (the first part of RQ1), a general methodology for identifying praxeological differences (the second part of RQ1) will be outlined and then applied to examine the Danish case.

The first step in identifying praxeological differences between two connected institutions is to construct a common praxeological reference model (PRM). The PRM serves to classify the mathematical content taught and learned across the institutions, and to make the praxeological differences visible by describing them in terms of the model.

8.1.1 Construction of a praxeological reference model for secondary school algebra

As presented in Chapter 2.3, a PRM for a mathematical domain algebra at the secondary level is constructed through a praxeological analysis of the algebraic praxeologies found in curricular documents, resources, official test material, textbooks, and teaching material. These resources provide the opportunity to observe which types of tasks, techniques, notions, properties, and results constitute the knowledge to be taught for a specific mathematical domain, as illustrated in the Spanish case by Barbé et al. (2005), which has inspired the development of a PRM for secondary-level algebra.

For the Danish case, the PRM for algebra at the secondary level was developed through a study along this line of both lower and upper secondary schools. As for lower secondary school, we relied on official documents – especially the centralized exams – that provide knowledge about what students are supposed to be able to do at the end of ninth grade (the year before starting upper secondary school). For this purpose, the ninth-grade final official exam (an evaluation instrument) and results for students' performance were used to gain knowledge about the algebraic content taught and expected to be learned after lower secondary school. Similarly, so-called screening tests serve as documents from upper secondary schools to identify what they are expected to know there. These screening tests, which the Danish Government required schools to use between 2017 and 2019 to assess their students after two months, consist of eight STX tests (The Higher General Examination) with a primary focus on linear functions and regressions. These eight tests were used in the development of the praxeological model, insofar as the tasks required algebraic knowledge (Paper 1, p. 7).

Additionally, introductory chapters from upper secondary school textbooks were included in the praxeological analysis to derive concrete information about what students are expected to know at the entrance. Official curricula for lower secondary school were not useful in this study as they are too vague regarding concrete mathematical contents and only provide recommended rather than mandatory learning goals (Paper 1, p. 7).

The model was developed by identifying algebraic praxeologies in these documents, resulting in a praxeological reference model for secondary-level algebra in Denmark as a whole. The detailed model is presented in Chapter 9.1, and consists of three local algebraic organizations (AO)

- AO_1: Setting up an algebra model based on numerical information
- AO_2: Substituting into an algebra model
- AO_3: Rewriting or operating on an algebraic model

(Paper 1, p. 6)

Although the model is presented through three local algebraic organizations, each organization may comprise different techniques and technological elements depending on the institution. A model developed for one institution alone can therefore not serve as a reference model for classifying praxeologies from another institution. The praxeological reference model is accordingly developed on the basis of a praxeological analysis of documents from both lower and upper secondary schools. In particular, AO_3 includes a variety of techniques for rewriting or operating on algebraic expressions, which are known to be organized differently in lower and upper secondary school, as elaborated on in Chapter 9.1.

8.1.2 How to identify praxeological differences?

In Chapter 2.2, praxeological differences were expressed as $MO^{I_2} \setminus MO^{I_1}$, where MO^{I_2} refers to the mathematical organization students are expected to have mastered when entering a new institution I_2, and MO^{I_1} refers to the mathematical organization that a certain share of the students actually learned in the previous institution I_1. As pointed out in Paper 1, the "certain share" "must be fixed and justified according to the context and aims of a given study; it could, for instance, be the majority of those entering I_2" (p. 4).

To determine this praxeological difference, one must first identify MO^{I_2} and then examine whether the elements found in MO^{I_2} can also be found in MO^{I_1}. This procedure presupposes a praxeological reference model that allows for the classification of praxeologies from both institutions. Algebraic praxeologies expected in I_2 may already be present in I_1, but organized differently with respect to types of tasks, techniques, or technological elements. This is the case, for example, for tasks related to solving first-degree equations. In Danish upper secondary school, such tasks are classified within AO_3, as they typically require rewriting and operating on the equation in order to solve it. In lower secondary school, tasks involving first-degree equations may instead be classified within AO_2, as the equations can be solved using a substitution technique. These aspects will be elaborated in Chapter 9.1.2.

The use of a praxeological reference model developed on the basis of documents from both institutions, therefore, allows such praxeologies to be identified and compared within a common model. This procedure will be illustrated in the Danish case. These mathematical organizations considered are those identified in the praxeological analysis and must be fully covered by the praxeological reference model, in the sense that praxeologies from both institutions can be classified in their entirety within the model.

We use the obvious notation $AO_n^{I_2} \setminus AO_n^{I_1}$ when talking about the algebraic praxeologies. Concretely, the praxeological differences between Danish lower secondary and upper secondary schools can be identified as the union of $AO_1^{I_2} \setminus AO_1^{I_1}$, $AO_2^{I_2} \setminus AO_2^{I_1}$ and $AO_3^{I_2} \setminus AO_3^{I_1}$ (Paper 1, p. 7).

To determine $AO_n^{I_2} \setminus AO_n^{I_1}$ for $n = 1,2,3$, the first step is to determine $AO_n^{I_2}$ for $n = 1,2,3$. This is done by analyzing the aforementioned documents, including the introductory chapters of textbooks used at the beginning of upper secondary school, as well as the previously mentioned Screening Test.

Although tasks related to the three algebraic organizations are readily identified through the analysis of introductory chapters in textbooks, these are not necessarily representative of all schools. Different books are used in different upper secondary school programs, and the tasks do not only indicate what students should master in algebra. They may also include tasks that indicate what students are expected to learn at the beginning of upper secondary school.

To determine whether these tasks relate more to knowledge to be developed or knowledge students are expected to already master, one can analyze "the level of detail in the examples presented in the textbooks [with] a careful examination of the specificity and thoroughness with whish an example is written or explained can explicitly reveal what students are expected to already know to comprehend the examples, as well as what new concepts are introduced therein" (s. 7).

The next step in identifying $AO_n^{I_2} \setminus AO_n^{I_1}$ for $n = 1,2,3$, is to determine whether the elements identified for $AO_n^{I_2}$ for $n = 1,2,3$ are also present in $AO_n^{I_1}$ for $n = 1,2,3$.

To determine $AO_n^{I_1}$ for $n = 1,2,3$ for the Danish lower secondary school, the final exams for ninth grade and students' performance on these exams are used. This analysis considers both whether a type of task appears in I_1 and how representative it is in terms of frequency. It also examines how many students can actually solve the tasks, even when they appear in the final exam. For the analysis of Danish lower secondary school exams, items from a total of 21 exams from 2018 to 2023 were reviewed. Each exam consists of two parts: one allowing aids and one without. Both parts are included in the analysis.

Paper 1 points out that fixing the aforementioned "certain share" can be challenging. In Denmark, roughly 70% of students proceed from lower to upper secondary school, making 70% a reasonable benchmark. In practice, however, this share is more difficult to fix, as task prevalence must also be considered (p. 8).

8.2 Methodology for RQ2 and RQ3

The methodology is organized into three parts. First, the experimental design is presented, including how the bi-institutional lesson study was conducted, who the participating teachers and other actors were, and the roles they assumed throughout the process. The selection of schools and teachers is also described. Second, the qualitative methods used to investigate RQ2 and RQ3 will be described in a single chapter, as the same methods, such as observations, interviews, and the use of lesson plans, were applied for both questions. Third, we explain how the data generated through these methods was analyzed, presented in two separate chapters corresponding to RQ2 and RQ3.

8.2.1 The experiment

The bi-institutional lesson studies were conducted over the school year 2023/2024, involving teachers from both lower secondary and upper secondary schools within the same municipality, where students typically transition directly from the lower secondary school to the upper secondary school.

The project began with a workshop focusing on three areas. First, a professor presented lesson study from a research perspective. Second, a researcher, who is also a practitioner, shared his experience with lesson study in primary and lower secondary schools in Denmark. Finally, the two participating upper secondary school teachers presented their experiences from conducting lesson study in the upper secondary school.

Initially, six lower secondary school teachers and two upper secondary school teachers participated; however, due to employment changes, the number of lower secondary school teachers was reduced to five. The upper secondary school teachers had prior experience with lesson study, which was considered an advantage but not a requirement.

Each lesson study followed a structured process (For an overview, see Fig. 1). It began with a problem/task selection meeting, where one lower secondary school teacher (the one who also teaches the lesson) and one upper secondary school teacher studied and discussed the curriculum, teaching materials, and their experiences with algebra at their respective institutions. They also identified potential student difficulties during the transitions between the institutions. Based on this discussion, they chose a type of task for the research lesson. The curriculum and the teaching materials considered included the ninth-grade final exam, textbooks for the ninth grade, and the screening test items used in the upper secondary school.

This was followed by two two-hour planning meetings, where all teachers collaborated to formulate or refine the problem, solve it themselves to anticipate potential students' responses and misconceptions, and complete a detailed lesson plan (Template translated from Bahn, 2018, Appendix F; illustrated in Appendix 1, Chapter 13.1). Learning goals for the lesson were formulated during these discussions. I attended these meetings as an observer, occasionally supporting the teachers by reminding them of the meeting objectives or helping to articulate aspects of the lesson plan.

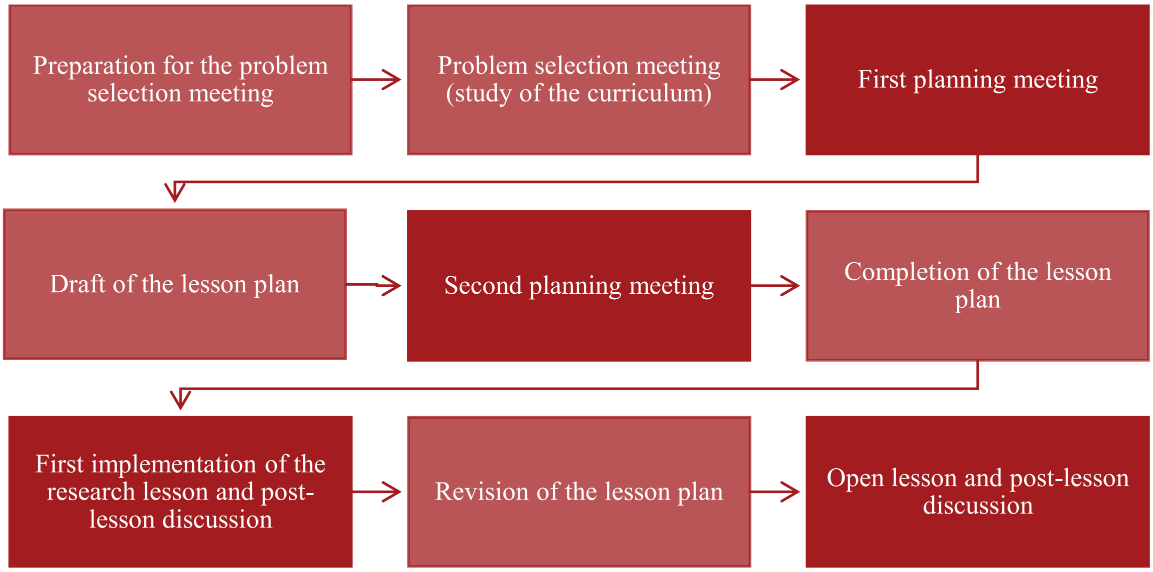

Fig. 1: Overview of the bi-institutional lesson study process. Figure copied from Paper 2

Over the year, each of the two teams planned and conducted three bi-institutional lesson studies. In addition, all teachers observed the other team's open lessons, so that each teacher participated in a total of six research lessons. These are referred to as LS1, LS2, LS3, LS4, LS5, and LS6, with the first three designed by team one and the last three by team two. Each lesson study lasted about five weeks, and the teams took turns conducting the lesson studies.

Each lesson study involves a first implementation of the research lesson, involving only the team who had planned the lesson and me. These sessions served to try out the plan and reflect on any misunderstandings, particularly regarding the problem formulation. I facilitated the post-lesson discussions. The first implementation was followed by the main research lesson, observed by both teams. A knowledgeable other and a facilitator were also present.

During the subsequent post-lesson discussion, the teacher, who taught the lesson, first shared their experience of the lesson, including what had gone particularly well and what challenges had arisen in relation to the anticipations of the lesson plan. The rest of the team and other observers then contributed with concrete observations, which were noted by the facilitator. Emerging themes from the facilitator's notes were discussed in detail. The facilitators had extensive practice and research experience in lesson study.

The meeting was concluded with final comments from the knowledgeable other, either a professor from the university or a researcher from the university college, who connected the observations to research and provided concrete suggestions for improving algebra instruction.

Each lesson study concluded with a lesson report, co-written by the lower secondary school teacher that taught the lesson and the upper secondary school teacher, based on a provided template (Template from Østergaard et al., 2020; also reproduced in Appendix 2, Chapter 13.2).

8.2.2 The qualitative methods

The study draws on qualitative data collected throughout the implementation of the bi-institutional lesson study. The qualitative methods consist of observation notes and audio recordings from all planning meetings and post-lesson discussions, which were manually transcribed, as well as documents (lesson plan, handouts from study lesson) and pictures of students' and teachers' writing during the lesson.

As a researcher, I adopted a facilitator role in the collaborative work with teachers. I participated in planning meetings and post-lesson discussions that constituted both the context for the study and the primary source of data. My role was to facilitate the process by ensuring that meetings followed the agenda and time frame, and by supporting the continuity of the collaboration. While observing and documenting teachers' practices, I did not take part in decisions regarding task selection, lesson planning, or instructional design. On occasions when teachers explicitly asked for my perspective, I offered limited suggestions intended to support the discussions rather than to direct them. I positioned myself as a facilitator of the collaborative process while collecting data as a researcher. The teachers were informed about and aware of my dual role as facilitator of the planning meetings and as researcher collecting data, which contributed to maintaining transparency regarding my involvement.

Additionally, a semi-structured interview with each participating teacher was conducted at the end of the project. This format was selected because it allowed me to address specific, predetermined questions while simultaneously providing the opportunity to explore emergent issues and topics identified by the teachers themselves. Furthermore, the semi-structured approach facilitates in-depth follow-up discussions, enabling the researcher to capture and elaborate on the aspects that teachers consider important (Brinkmann, 2018, p. 579).

The interviews were divided into two parts. The first focused on eliciting teachers' self-reported learning outcomes and challenges related to the bi-institutional lesson study process, with questions such as *"Can you describe a moment during the year that you found particularly meaningful or valuable?"*, *"Can you recall a situation that felt particularly difficult?"*, *"Did you experience any obstacles during the planning or implementation of the bi-institutional lesson study?"*, and *"How has it been to work with algebra? Can you describe situations where this became evident?"*. The second part explored how the collaboration with teachers from neighboring institutions was experienced, for example, through questions like *"Can you describe a situation where the collaboration with the other institutions was especially meaningful and productive?"* and *"Were there moments where the collaboration felt challenging?"*. Together, these data form the basis for the identification of what conditions and constraints that appear to facilitate the establishment of the bi-institutional lesson study and the analysis of teachers' development of practical and theoretical didactic knowledge about the transition problem in algebra.

8.2.3 Analysis of qualitative data for RQ2

As outlined in Chapter 7, the work presented in paper 2 addresses RQ2: “What conditions and constraints appear to facilitate the establishment and process of bi-institutional lesson study?”. The empirical data for this analysis consists of observation notes and audio recordings from planning meetings and post-lesson discussions, which were subsequently transcribed, as well as documents (lesson plan, handouts from study lesson) and pictures of students’ and teachers’ writing during the lesson.

The analysis for RQ2 focused particularly on Team 2’s first lesson study. Since RQ2 focuses on the establishment of a new paradidactic infrastructure, it is reasonable to examine the early stages of the collaboration, which could be moments where the teachers collaborate for the first time with teachers from neighboring institutions with whom they usually neither meet nor collaborate. For this reason, the data analysis for RQ2 is based on Team 2’s first bi-institutional lesson study. Data from Team 2 aligns with the data from Team 1, making this focus unproblematic.

The first step in the data analysis was to read the transcriptions from start to finish to gain an overall view of the collaboration and interaction between the teachers. In a second reading, episodes or moments were identified. These episodes were then grouped according to thematic relevance, such as teachers’ expectations of students, discrepancies between anticipated and observed student performance, and the teachers’ knowledge of algebra and teaching of algebra. Throughout this process, I focused on episodes where the teachers’ didactic knowledge developed, to examine how the paradidactic infrastructure enabled or constrained this development.

As described in the methodology section in Paper 2 (p. 9), the critical moments within these episodes were identified according to whether they were:

- episodes where the development of shared theory blocks seems to advance in critical ways, such as building common justifications of didactic praxis through sharing and combining knowledge from the two institutions
- episodes where the teachers seem to run into critical obstacles, for example, due to different norms or assumptions, related to the two institutions
- episodes where the teachers’ discussions were explicitly related to algebraic concepts and tasks.
- episodes where different interpretations or misunderstandings arose among the teachers due to a lack of a common model for algebra
- episodes, where discrepancies arose between the teachers’ expectations of students’ performances and the actual student behavior.

(Paper 2, p. 9)

By carefully analyzing these episodes and moments, it becomes possible to identify the conditions and constraints that influenced the establishment of the bi-institutional lesson study.

8.2.4 Analysis of qualitative data for RQ3

As outlined in Chapter 7, the work presented in Paper 3 and Paper 4 addresses RQ3: "In view of reducing specific praxeological differences in algebra (as identified in RQ1), how does bi-institutional lesson study function as a paradidactic infrastructure to develop relevant didactic praxis and theoretical knowledge related to these praxeological differences?".

The empirical data for this analysis consist of observation notes and audio recordings from planning meetings, post-lesson discussions, and interviews with teachers, which were subsequently transcribed, as well as documents (lesson plan, handouts from study lesson) and pictures of students' and teachers' writing during the lesson. The methodology for analyzing these data is the same as the one detailed in Paper 3, so this section will briefly describe how the data analysis is conducted for examining RQ3 and point the reader to Paper 3 for more details.

The analysis was carried out in three main phases. First, all data were reviewed and sorted according to their relevance to the transition problem in algebra or to the bi-institutional lesson study, with attention to episodes indicating the development of mathematical or didactic praxeologies. Second, the interview data were examined to identify situations in which teachers described the collaboration as difficult or beneficial. Third, transcripts from the planning meetings and post-lesson discussions were analyzed alongside the transcripts of the interviews and the lesson plans for the research lesson. This made it possible to compare the teachers' didactic intentions with their subsequent reflections and discussions, and to explore how these interactions supported or challenged the development of their didactic praxeologies and theoretical knowledge related to the praxeological differences.

9 Results

9.1 Main results for RQ1

9.1.1 Praxeological reference model for secondary school algebra

Following the methodology presented in Chapter 8.1.1 (and Paper 1), a praxeological analysis of relevant material from Danish lower and upper secondary schools is conducted. This analysis resulted in a praxeological reference model (PRM) for school algebra at the secondary level (Table 1). The PRM consists of three algebraic organizations – AO_1, AO_2, and AO_3 – each comprising specific types of tasks and corresponding techniques.

AO_1: Set up an algebraic model	AO_2: Substituting in an algebraic model	AO_3: Rewrite (operate on) an algebraic model
$T_{1,1}$: Set up a first-degree equation based on a verbal description with numerical data. $T_{1,2}$: Set up an algebraic model based on a geometrical situation, usually involving a diagram with symbols attached.	$T_{2,1}$: Substitution of numbers into a linear equation. $T_{2,2}$: Substitution of numbers into a given algebraic expression.	$T_{3,1}$: Rewrite (operate on) a first-degree equation. $T_{3,2}$: Rewrite (operate on) an algebraic expression

Table 1. Praxeological reference model for school algebra at the secondary level in Denmark (copied from Paper 1).

The PRM aims to encompass algebra as it appears across both lower and upper secondary school, thereby serving as a shared reference model for analyzing students' algebraic knowledge across institutions.

The PRM shares characteristics with conceptualizations of school algebra, as presented in Chapter 5. The algebraic organization AO_1(setting up an algebraic model) corresponds to Usiskin's (1988) algebra as problem-solving procedures and with Kieran's (2004) generational activities, both of which emphasize the development of algebraic representations of a situation. AO_2, focused on substitution, can be seen as a specific example of procedural thinking as described by Kieran (1992), which more broadly involves performing arithmetical and algebraic operations. AO_3 involves rewriting and operating on algebraic models and first-degree equations without substituting numerical values. In Kieran's (1992) terms, this reflects structural thinking, as the focus is on the structure and equivalence of algebraic expressions rather than on numerical results. Moreover, AO_3 aligns with Kieran's (2004) transformational activities since the operations preserve algebraic relationships through transformations, such as simplifying algebraic expressions or rewriting equations.

What distinguishes the PRM from these earlier descriptions of school algebra is its praxeological specificity. Whereas Kieran's categories may describe students' difficulties at a conceptual level, for example, difficulties with transformational activities, the PRM makes it possible to determine whether such difficulties belong to AO_2 or AO_3, depending on the techniques employed by the students (e.g., substitution in AO_2, where only one technique exists, or algebraic transforming and rewriting in AO_3, where many techniques exist). In the PRM, AO_1, AO_2, and AO_3 are large algebraic organizations consisting of many types of tasks, each characterized by particular techniques, and in the case of AO_3, by the existence of a variety of algebraic techniques, as described by Poulsen (2015). Thus, solving equations may fall under either AO_2 or AO_3, depending on how students actually solve a task. This level of detail and praxeological specificity in the PRM makes it possible to locate students' difficulties at the level of specific techniques within an algebraic organization, rather than at a broad conceptual level. The PRM, therefore, allows for fine-grained analyses of students' algebraic praxeologies. This specificity can, for instance, support the development of diagnostic tests (Cosan, 2021) that can identify which techniques students master, lack knowledge of, or experience difficulties with. Such precision in the PRM is not found in the more conceptual characterisations of school algebra in the literature.

Connections to the literature about school algebra within ATD are also to be found. $T_{3.1}$ and $T_{3,2}$ in AO_3 in Table 1 corresponds to the techniques associated with stages 1 and 2 in Ruiz-Munzón et al.'s (2013) reference epistemological model, where arithmetic praxeologies become algebraic through the manipulation of algebraic expressions (stage 1) and the manipulation of identities and equations (stage 2). The focus on creating algebraic expressions in stage 1 corresponds to AO_1. The PRM shares with Ruiz-Munzón et al. (2013) this focus on techniques for transforming and rewriting algebraic expressions and equations, corresponding to stages 1 and 2 of algebraization. Unlike their model, which explains how algebra develops from arithmetic, the PRM specifies which algebraic techniques are present in Danish secondary level. Thus, the PRM does not model algebraization itself but captures how its early stages are enacted in secondary school in the Danish context.

The PRM is also related to the dual role of algebra described by Bolea et al. (2004). According to Bolea et al. (2004), students' early encounters with algebra are grounded in arithmetic and contribute to a viewpoint of algebra as generalized arithmetic, where algebra functions as a symbolic system that expresses arithmetical rules in general terms. At the same time, algebra can be understood as a modelling tool within a broader algebraization process, where algebra is used to model and justify activity in other mathematical domains and where letters represent quantities within relations. Elements of both perspectives can be identified in the PRM; the types of tasks associated with AO_1 and AO_2 reflect algebra as generalized arithmetic through the construction and use of symbolic expressions based on numerical relationships, while components of AO_1 and AO_3 connect to algebra as a modelling tool, when algebraic structures are used to represent relations or justify problem-solving processes beyond arithmetic alone.

In this way, the PRM is positioned in close relation to central characterisations of algebra as generalisation, manipulation, and modelling (Usiskin, 1988; Kieran, 1992, 2004, 2022; Bolea et al., 2004). At the same time, it extends these by grounding algebraic activity in specific types of tasks and techniques rather than in conceptual categories alone.

Although previous literature on school algebra, both within ATD and beyond, provides important knowledge about the background and nature of algebraic activities, these descriptions emphasize broad types of reasoning and uses of variables, rather than the concrete tasks and the corresponding techniques through which algebra is enacted among students. As a consequence, such descriptions offer limited support for analyzing where, how, and why specific algebraic difficulties emerge in practice. The praxeological reference model in Table 1 can be used to address this by further specifying the tasks and techniques that shape algebra in the Danish context. An example of such use is provided by Cosan (2021), who develops a diagnostic test based on a praxeological reference model to identify students' mastery of, and misconceptions with, arithmetical and algebraic techniques.

Additionally, constructing a new PRM was necessary because the present research concerns the transition between two neighboring institutions. A reference model grounded solely in upper secondary school would not capture the algebraic knowledge and expected knowledge among the students at the end of lower secondary school, and vice versa. The PRM, therefore, had to be built upon materials and practices from both institutions to enable a meaningful analysis of the transition between them, which will be elaborated upon in the following chapter.

9.1.2 Praxeological differences in Danish secondary level

The purpose of this section is to present how praxeological differences between Danish lower secondary school (LSS) and upper secondary school (USS) relate to the transition problem in algebra. The analysis below shows that AO_1 and AO_2 are almost aligned across the two institutions in terms of the types of tasks students encounter and the techniques expected to be mastered. By contrast, the praxeological differences related to AO_3 is not empty and is therefore central to explain the transition problem.

The empirical data, also elaborated in Paper 1, indicate that $\mathrm{AO}_1^{\mathrm{USS}} \setminus \mathrm{AO}_1^{\mathrm{LSS}} \approx \emptyset$ and that $\mathrm{AO}_2^{\mathrm{USS}} \setminus \mathrm{AO}_2^{\mathrm{LSS}} \approx \emptyset$ while $\mathrm{AO}_3^{\mathrm{USS}} \setminus \mathrm{AO}_3^{\mathrm{LSS}} \not\approx \emptyset$. This suggests that the transition problem is primarily related to AO_3 .

Tasks belonging to AO_1, such as setting up linear models from written, verbal descriptions or geometrical situations (e.g., $\mathrm{T}_{1,1}$ and $\mathrm{T}_{1,2}$) and AO_2, such as substituting numerical values into linear models or algebraic expressions, occur with similar frequency in lower secondary and upper secondary schools. Students in both institutions identify variables and parameters and set up linear equations like $y = ax + b$. For example, a comparison of a task from the Screening test from 2017 from upper secondary school and a task from the final ninth-grade exam in 2023 for lower secondary school shows that students work with equivalent types of tasks and techniques related to AO_1. Thus, tasks related to AO_1 occur in both institutions with more or less the same prevalence, and at the end of lower secondary school, slightly more than half of the students can solve such tasks correctly (Paper 1, p. 11).

Similarly, it is expected that at the beginning of upper secondary school, the students can solve tasks related to AO_2, where the main technique is substitution into a function, for example, substituting $x = 5$ into $y = 2x + 3$. Tasks related to AO_2 also occur in lower secondary school, where students often solve first-degree equations using a substitution technique. For example, in lower secondary school, equations often have positive integer solutions, making backward reasoning ("what must x be to satisfy the equation?") and substitution effective. Crucially, it is the techniques required to solve the task, not the solution itself, that determine whether a task belongs to AO_2 or AO_3. Some first-degree equations may belong to AO_3, in practice, yet can still be solved using techniques related to AO_2. Therefore, tasks related to AO_1 and AO_2 exist across institutions and do not significantly contribute to the transition problem (Paper 1).

By contrast, tasks belonging to AO_3 require algebraic transformation or rewriting rather than numerical substitution. AO_3 includes tasks such as rewriting and transforming algebraic expressions and first-degree equations using algebraic techniques, e.g., the distributive law, operating on both sides of an equation or expression, handling parentheses, negative numbers, fractions, and exponent rules (Paper 1).

A typical task from upper secondary school related to $T_{3,1}$ in AO_3 is

$$3x + 2(x + 1) + 7 = 5$$

$$3x + 2x + 1 + 7 = 5$$

$$5x + 8 = 5$$

$$5x = 3$$

$$x = -\frac{5}{3}$$

The task for the students is to identify and describe the mistakes that are made in the rewriting (Paper 1, p. 12). Such tasks require students to work with algebraic techniques that go beyond substitution (Paper 1).

In lower secondary school, a few tasks related to AO_3 requiring students to carry out such algebraic rewriting themselves. When tasks related to AO_3 appear in lower secondary school, they often ask students to explain someone else's rewriting rather than perform the rewriting. For example, exercise 6.3 in the ninth-grade final exam 2021 (Paper 1, p. 14) had only 20% correct responses, which illustrates that techniques related to AO_3 are not established before upper secondary school (Paper 1).

Every year, three first-degree equations appear in the ninth-grade final exam. In 2023, the three equations were:

- 7.1: $6x + 5 = 41$
- 7.2: $4 \cdot (x + 1) = 5x$
- 7.3: $\frac{x}{2} + 12 = 2x - 3$

All three can be solved using a backward reasoning and substitution technique, since the equations are structured so that students can find integer solutions systematically. Although the equations differ in complexity, backward reasoning and substitution are sufficient to solve all three. The variation in students' success rates across the three equations (approx. 80%, 47%, and 29%) (Paper 1, p. 11) suggests that many students' techniques remain arithmetic rather than algebraic. If the equations were solved using algebraic manipulation instead of backward reasoning, guess-and-try, and substitutions, the same techniques in AO_3 could be used to solve all three equations, but in practice, techniques related to AO_2 suffice. However, when students rely on guess-and-try and substitution techniques, their arithmetical techniques, such as calculating a fraction with an integer, make it difficult to solve the equation correctly (Paper 1).

This illustrates that tasks related to AO_3 is almost absent in lower secondary school, whereas it is required in upper secondary school, making $AO_3^{USS} \setminus AO_3^{LSS}$ the core praxeological difference and thereby the transition problems between the two institutions.

The praxeological differences related to AO_3 can be better understood through existing theoretical perspectives on students' transition from arithmetic to algebra. These perspectives help explain why techniques that are sufficient in lower secondary school become inadequate in upper secondary school, thereby clarifying the roots of the transition problem.

Filloy and Rojano (1989) (also presented in Chapter 5.2) describe a cut point between arithmetic and algebra, where techniques that were previously sufficient, primarily substitution of numbers into the first-degree equations, no longer suffice, and students must instead rewrite or transform algebraic expressions and equations. In lower secondary school, students typically work with equations of the form $Ax + B = C$ or $Ax + B = Cx$. For these forms, where guess-and-try, backwards reasoning, and substitution are often sufficient techniques to identify the solution, meaning that students are not required to cross the cut point.

Importantly, two equations may look similar in form but belong to different algebraic organizations (AO_2 or AO_3), depending on which technique is required to solve them. For example, the equation $6x + 5 = 41$ (task 7.1 from ninth-grade exam) has the form $Ax + B = C$ and can be solved using backward reasoning and substitution, classifying it as a task related to AO_2. By contrast, the equation $-3 \cdot x = 5$, which students are expected to solve upon entering upper secondary school, has the same form but requires algebraic rewriting, and therefore belongs to AO_3.

Similarly, tasks 7.2 ($4 \cdot (x + 1) = 5x$) and 7.3 $\left(\frac{x}{2} + 12 = 2x - 3\right)$ from lower secondary school formally have the structures $A(x + b) + B = Cx$ and $Ax + B = Cx + D$. Nevertheless, the techniques used in lower secondary school (backward reasoning, guess-and-try, and substitution techniques) are sufficient, and these tasks are therefore classified as AO_2, even though their algebraic form would, in principle, allow techniques related to AO_3. Furthermore, the equations students are expected to solve in lower secondary school, typically have positive integer solutions in the interval $\{1, \dots, 10\}$ and positive integer coefficients. These numerical properties do not in themselves determine whether a task belongs to AO_2 or AO_3. This classification is still determined by the algebraic form of the equation and the techniques required to solve it. However, they establish conditions under which techniques related to AO_2, such as backward reasoning and substitutions, are both possible and sufficient. Because these tasks are solved without aids in the exam, it is not possible to determine exactly which techniques students actually apply. However, the fact that they can be solved without algebraic rewriting, using algebraic rules, indicates that students are not required to cross the cut points in lower secondary school.

In upper secondary school, however, equations of similar forms, such as $Ax + B = Cx + D$, often require algebraic transformation or rewriting to be solved. Here, backward reasoning, guess-and-try, and substitution may no longer be sufficient, and students must operate symbolically on both sides of the equal sign in the equation. The key difference is therefore the required technique rather than the type of solution.

In this sense, the praxeological differences related to AO_3 therefore aligns with Filloy and Rojano's (1989) cut point: students leave lower secondary school equipped with techniques that remain below the cut, but upper secondary school demands techniques situated beyond it.

Kieran's (1981) work further explains why the shift from AO_2 to AO_3 becomes difficult. Students aged 12-14 often interpret the equal sign operationally, as an instruction to "do something next", rather than relationally, where the equal sign expresses an equivalence between two quantities. In lower secondary school, where the backward reasoning, substitution, and guess-and-try techniques dominate, such an operational interpretation is sufficient and aligns well with tasks related to AO_2. For instance, when students solve the equation $6x + 5 = 41$ in problem 7.1, by thinking "what should x be so $6x = 36$". This approach is not possible with equations from upper secondary school such as $3(14 + x) = 9$ (Paper 1). By contrast, tasks related to AO_3 in upper secondary school require a relational understanding of the equal sign, since students must perform operations on both sides of the equation to isolate the variable. Kieran (2022) also distinguishes between analytical, structural, and functional algebraic thinking. Techniques in AO_3 correspond to structural and functional thinking, yet lower secondary school students predominantly work analytically with concrete numbers, further reinforcing the transition problem.

The difficulty students encounter when unknowns appear on both sides of an equation can also be described as a didactic cut, following Herscovics and Linchevski (1994). In lower secondary school, students can rely on backward reasoning, substitution, and guess-and-try techniques, which allow them to find solutions without fully engaging in symbolic manipulation or algebraic rewriting, even when variables appear on both sides. However, these techniques become insufficient at the upper secondary school. For example, in equations such as $3(14 + x) = 9$, isolating the unknown requires manipulation of algebraic expressions, a technique that students have not previously needed. Consequently, students experience a didactic cut precisely when AO_3 techniques become indispensable, as the previously sufficient techniques related to AO_2 are no longer useful.

From the perspective of ATD, tasks related to AO_3 mark the point where arithmetic praxeologies may be algebraized. Lower secondary school practices mostly correspond to stage 1 (simplification of concrete values), whereas upper secondary school requires progression into Stages 2-3, where students manipulate symbolic expressions and parameters. This reinforces AO_3 as a central praxeological cut point.

Taken together, these theoretical perspectives show that the praxeological differences related to AO_3 are not merely empirical, but correspond directly to a mathematical and didactic discontinuity. Most students leave lower secondary school with techniques sufficient for AO_2 but insufficient for AO_3, indicating that most have not crossed this cut point before entering upper secondary school. This can help explain why the transition problem emerges precisely where algebraic manipulation becomes necessary in equation solving.

It is important to emphasize that the transition problem identified here does not stem from unfounded expectations in upper secondary school. In fact, the presence of three first-degree equations in every ninth-grade final exam, all with positive integer solutions between 1 and 10, implicitly shows that students are expected to enter upper secondary school able to solve such tasks. However, because these equations can be solved through backward reasoning, guess-and-tr and substitution techniques, students are not required to apply the algebraic techniques needed to manipulate algebraic expressions. As a result, students arrive at upper secondary school having encountered tasks, such as solving first-degree equations, which do not require techniques beyond AO_2 and therefore do not require students to cross the algebraic cut point or the didactic cut described by Filloy and Rojano (1989) and Herscovics and Linchevski (1994). Therefore, the transition problem arises not from excessive demands by upper secondary school teachers, but from a concrete praxeological difference related to AO_3: students have not been required to master algebraic manipulation in lower secondary school, yet this mastery becomes necessary upon entering upper secondary school.

9.2 Main results for RQ2

Following the methodology presented in Chapter 8.2.3 and the analysis reported in Paper 2, the following Chapters will present the main results related to RQ2: What conditions and constraints appear to facilitate the establishment and process of bi-institutional lesson study?

The results are discussed in relation to existing literature on lesson study in Japan (Chapter 4.1) and lesson study beyond Japan (Chapter 4.2).

9.2.1 A common model in algebra

Bi-institutional lesson study is particularly relevant when concrete and urgent transition problems exist between two neighboring and connected institutions. In the present case, the transition problem concerned praxeological differences in school algebra between Danish lower secondary and upper secondary schools. A key finding from Paper 2 is that the establishment of the bi-institutional lesson study presupposes the development of a shared model for the subject matter that constitutes the transition problem between the two institutions.

Because the teachers have different backgrounds and experiences with students and the algebraic content, they cannot assume a shared view of what algebra entails or where the students' difficulties are located. Without an explicit common model, the teachers' discussions can remain vague or rely on implicit and institution-specific knowledge. The results, therefore, indicate that the establishment of a common model is a necessary condition for making the transition problem explicit and discussable across the two neighboring institutions (Paper 2).

As presented in detail in Paper 2, the common model in algebra constructed by the teachers did not take the form of an explicitly articulated praxeological reference model, such as the one presented in Table 1 (in Chapter 9.1) with clearly specified types of tasks and techniques. Rather, it took the form of a rough common model in which school algebra was distinguished into three types of task: 1) setting up an algebraic model, 2) substituting into an algebraic model, and 3) rewriting or operating on an algebraic model (Paper 2). While not formulated in terms of praxeologies, these distinctions can be seen as corresponding to the algebraic organizations presented in Table 1.

Despite its simplicity, this model worked as a shared reference. It enabled teachers to identify more precisely where students experienced difficulties, according to their experiences within both institutions, and to align their discussions around specific aspects of algebraic work.

In the planning meetings, the model supported task selection that explicitly addressed the transition problems and the praxeological differences. In the post-lesson discussions, it allowed the teachers to point concretely to where students have difficulties within the model and to discuss and develop possible didactic praxeologies to work with these difficulties (Paper 2).

From a lesson study perspective, the development of such a common model is closely connected to *kyozaikenkyu*, the in-depth study of the curriculum, particularly teaching materials. As highlighted in Chapter 4.2, insufficient attention to curriculum study is a well-documented challenge in lesson study implementations beyond Japan (Fujii, 2018; Seleznyov, 2018). The present findings suggest that *kyozaikenkyu* becomes even more critical in a bi-institutional lesson study context, where differences between curricula, expectations, and praxeologies must be negotiated explicitly. In this sense, while a common model emerges as a key condition for the bi-institutional lesson study, its development depends on a prioritization of curriculum study, which cannot be taken for granted in contexts beyond Japan.

9.2.2 Teacher interest and engagement as a condition

A second key finding concerns the role of teachers' interest and engagement as a condition in the establishment of the bi-institutional lesson study. The analysis in Paper 2 shows that while genuine interest in each other's practices is crucial for the collaboration, this interest was unevenly distributed between the teachers from the two institutions.

The upper secondary school teacher consistently expressed curiosity about the practices of lower secondary school teachers and explicitly positioned himself as a learner seeking to build on the experiences and expertise of the lower secondary school teachers (Paper 2). By acknowledging the lower secondary school teachers as experts on their students' needs and difficulties, he contributed to establishing a respectful and open collaborative relationship. This stance supported knowledge exchange across the institutions and created conditions for shared reflection (Paper 2).

By contrast, the lower secondary school teachers showed limited interest in the practices of upper secondary school (Paper 2). This can be attributed to the fact that the lesson study and research lesson were conducted in a lower secondary school. Their contributions were primarily oriented toward the research lesson, such as how to present and introduce the task, anticipating and focusing on students' responses, including correct and incorrect solutions. While this focus is understandable, given that the research lesson was conducted in a lower secondary school, it also constrained the potential for mutual interest and learning. Even when the upper secondary school teacher shared experiences about students' difficulties in algebra at upper secondary school, these contributions were rarely followed up with questions or reflections from the lower secondary school teachers (Paper 2).

This asymmetry in interest and engagement can constitute a structural constraint for bi-institutional lesson study. Since the purpose of the collaboration is to address transition problems and praxeological differences between institutions, attention to both institutions would be expected. The limited attention to upper secondary practices in the enacted collaboration may indicate that this purpose was not made explicit or salient in the bi-institutional collaboration as carried out by the teachers. When interest is predominantly oriented toward one's own practice, opportunities for developing a shared knowledge about transition problems and praxeological differences can be reduced.

The importance of teacher interest and engagement is well-documented in the lesson study literature. Even in Japan, where lesson study is embedded in a well-established paradidactic infrastructure and supported institutionally through *konaikenshu*, participation relies on teachers' own commitment (Fernandez & Yoshida, 2004). In contexts beyond Japan, where lesson study lacks such institutional paradidactic infrastructure, teacher engagement becomes an even more important condition for the establishment and sustainability of the bi-institutional lesson study (Hart et al., 2011). The present findings above suggest that in bi-institutional lesson study, mutual and explicit interest, engagement, and commitment are not only desirable but a productive condition for the development of a new shared paradidactic infrastructure.

9.2.3 Roles and responsibilities in bi-institutional lesson study

A third key finding concerns the roles and responsibilities that emerge during the establishment of a bi-institutional lesson study. Although the research lesson and lesson plan were framed as collective products, in practice, responsibility was largely placed on the teacher who conducted the research lesson. While this may seem straightforward, it conflicts with the intended function of a lesson study plan as a collaborative product. Other teachers often directed questions to her and implicitly expected her to make key decisions regarding lesson planning (Paper 2).

The fact that only the teacher conducting the research lesson felt responsible for developing the plan constrained opportunities for collective decision-making and shared development of didactic and mathematical knowledge. Moreover, the teacher frequently assumed what can be described as a "secretarial role", focusing on documenting the lesson plan rather than actively participating in the didactic discussions. This highlights a tension in the planning process: although the lesson study plan is intended as a collective product that supports shared reflection and planning, in practice, it functions as an individual task. Consequently, the teacher's engagement in the collaborative planning, reflection, and discussion was reduced, despite her central role in the teaching (Paper 2).

An additional aspect of the roles in the bi-institutional lessons study concerns teachers' expectations of students' work. Anticipating how students will approach and work with a task, which techniques they may use, and which possible student responses are likely to emerge is a central element of lesson study and also an important part of teachers' didactic praxeologies. In the bi-institutional lesson study, lower secondary school teachers are the primary providers of such expectations, as they possess institutional knowledge of the students and their prior experiences (Paper 2).

However, the findings from Paper 2 show that even these well-founded expectations and hypotheses about hw students will solve a task do not necessarily align with students' actual performance in the research lesson. In particular, lower secondary school teachers were surprised by students' difficulties in setting up an algebraic expression involving a^2, despite expectations that this would be familiar from earlier work (Paper 2, p. 17). Thus, while lower secondary school teachers occupy a central role in anticipating students' answers and solutions in a bi-institutional lesson study, the empirical findings from Paper 2 show that this role does not guarantee correct anticipation. Rather, bi-institutional lesson study functions as an infrastructure in which teachers' assumptions about students' mathematical praxeologies in algebra are made visible, challenged, and potentially revised.

These findings highlight a key challenge in lesson study implementation outside Japan. In lesson study in Japan, roles and responsibilities are embedded within a paradidactic infrastructure, in which these roles are developed over time, progressing from novice participation to the ability to facilitate lesson study or act as a knowledgeable other. These roles are thus a part of the Japanese paradidactic infrastructure and supported by established norms, facilitators, and knowledgeable others (see Chapter 4.1 for details). In the establishment of the bi-institutional lesson study in Denmark, such infrastructure was absent, and roles therefore emerged implicitly and sometimes unevenly, which can create obstacles to both the establishment of the bi-institutional lesson study and the collaboration within it.

As emphasized in Paper 2, the Danish upper secondary school teachers, due to their academic background in mathematics and experience from teaching at the next level (p. 20), can offer new perspectives on concrete mathematical praxeology taught in lower secondary school, as well as new didactic hypotheses for teaching it (p. 20). This gives rise to reflection on the potential role as knowledgeable others. While it is expected that the upper secondary school teacher possesses strong content knowledge in algebra, due to their educational background (also described in detail in Chapther 6.1) this alone is insufficient to fulfil the role of a knowledgeable other as described by Takahashi (2014), which also requires the ability to connect theory and practice and to bring research-based knowledge into post-lesson discussion. In the Danish context, where sustained participation in lesson study is limited, such expertise is difficult to develop. The Danish upper secondary school teacher can therefore be better considered as a provider of content knowledge rather than as a knowledgeable other in the sense of the Japanese context.

At the same time, such content knowledge addresses a well-documented challenge in lesson study beyond Japan, namely, insufficient teacher content knowledge (Murata, 2011). Including upper secondary school teachers in a bi-institutional lesson study may therefore help to partially address this challenge. While these teachers could potentially be perceived as taking a more authoritative role due to their institutional background, the observed engagement and interest demonstrated by the upper secondary school teachers indicated that such hierarchical assumptions did not hinder collaboration (Paper 2). However, as the results indicate, more content knowledge alone does not guarantee an effectively functioning bi-institutional lesson study. Without clear, collectively negotiated roles and responsibilities, and without clarity about how the infrastructure supports them, the bi-institutional lesson study remains fragile.

9.3 Main results for RQ3

As described in detail earlier, the purpose of developing and working with bi-institutional lesson study as a new paradidactic infrastructure is to address concrete praxeological differences in algebra that are known to exist in the transition from Danish lower to upper secondary school. In particular, the discussion of the results related to RQ1 shows that the transition problem is closely related to praxeological differences related to AO_3, that is, tasks and techniques involving algebraic rewriting and transformation. Moreover, the analysis related to RQ2 identifies the conditions and constraints for establishing a bi-institutional lesson study as a paradidactic infrastructure across institutions.

Based on the findings from Papers 3 and 4, this section synthesizes how bi-institutional lesson study functions as a paradidactic infrastructure through which praxeological differences are made visible and worked with, and through which the development of teachers' didactic praxis and theoretical knowledge takes place. We focus here on key results that illustrate how teachers' didactic praxis and theoretical knowledge develop when bi-institutional lesson study explicitly addresses praxeological differences related to AO_3.

9.3.1 Making praxeological differences explicit through task design

Across Papers 3 and 4, bi-institutional lesson study is shown to function as a paradidactic infrastructure in which praxeological differences between lower and upper secondary school algebra become explicit through the collaborative design of tasks. Across all six bi-institutional lesson studies, the teachers shared the intention of designing tasks that required students to move from setting up an algebraic model (AO_1) to rewriting or operating on this model (AO_3). In this sense, task design became a means through which the algebraic transition problem identified in Paper 1 and described in Chapter 9.1 was addressed in practice.

This intention was, for example, clearly articulated in LS5 (the pocket money task) (described in Paper 3), where the teachers aimed to discourage substitution and guess-and-try techniques by constructing a task that was expected to require algebraic rewriting. As shown in Chapter 9.1, students predominantly rely on arithmetic and substitution and guess-and-try techniques. Observations from the research lesson, for instance, for LS5, confirmed this pattern: while most students continued to use these arithmetical techniques, only a small number of students engaged in algebraic manipulation as intended (Paper 3).

The tasks were based on algebraic content familiar to lower secondary school, such as geometrical situations or written descriptions, but were designed to support students' transition from arithmetic to algebraic techniques. Several tasks (e.g., LS1, LS2, and LS3 in Paper 3) were intentionally structured with a low entry point, enabling students to begin with concrete numerical values before being asked to generalize and express the situations algebraically. In these tasks, algebra primarily served as a means of modelling arithmetic, reflecting the conception of algebra as generalized arithmetic (Usiskin, 1988) (also described in Chapter 5.1).

At the same time, the discussion during task design and the reflection meetings within the bi-institutional lesson study highlighted how difficult it turned out to be to construct tasks that support the transition toward algebraic rewriting and manipulation. Across multiple bi-institutional lesson studies, teachers observed that while students could complete the initial arithmetic steps, many struggled to formulate algebraic expressions or carry out algebraic rewriting. A similar pattern emerged in the "think of a number" trick (Paper 4), where most students successfully completed the numerical part of the task but were unable to set up a correct algebraic model and simplify the algebraic expression.

These difficulties closely reflect the praxeological differences described in Chapter 9.1, not only in relation to students' algebraic praxeologies, but also in relation to the algebraic praxeologies embedded in teachers' didactic practices across both institutional levels. Thus, the arithmetic-to-algebra transition became visible both in students' work and in the didactic organization through which teachers designed, interpreted, and reflected on the algebraic tasks, aligning with the literature on the arithmetic-to-algebra transition. As Filloy and Rojano (1989) note, the shift from operating on concrete numerical values to operating on algebraic expressions represents a critical cut point. In the bi-institutional lesson study, this cut point became apparent not only in students' written work but also in teachers' planning meetings. Teachers recognized that, although the tasks were designed to allow rewriting using algebraic techniques, these techniques were not made necessary from the students' perspective (Paper 3).

The design process further revealed institutional differences in teachers' didactic organizations related to the algebraic organizations. Lower secondary school teachers tended to interpret the algebraic model as an abstraction of a concrete arithmetic procedure. This interpretation was particularly evident in the post-lesson discussion of the "think of a number" task analyzed in Paper 4, where the algebraic expression was described primarily as a generalization of the numerical steps performed by the students. From this perspective, algebra was predominantly considered as generalized arithmetic (Usiskin, 1988), corresponding primarily to algebraic organization related to setting up algebraic expressions (AO_1) and substitution in these algebraic expressions (AO_2).

Upper secondary school teachers, by contrast, focused more explicitly on the algebraic work required by the task, particularly on students' formulation of an algebraic expression to represent the trick and on the subsequent reduction of this expression to explain the outcome. Their attention was directed toward students' use of algebraic techniques, such as the distributive law, rather than toward the distance between the tasks and students' prior experiences. While lower secondary school teachers emphasized how unfamiliar this type of work was for the students, upper secondary school teachers focused on how students engaged with algebra (Paper 4).

These different perspectives did not hinder collaboration but instead became productive within bi-institutional lesson study, as they enabled teachers to articulate and negotiate their interpretations of students' algebraic work.

In Paper 3, it also became evident that tasks believed by teachers to pertain to AO_3 did not necessarily function as such in students' work. Rather, whether a task functioned as algebraic depended not only on its design, but also on students' prior experiences with such tasks and on the techniques that students were used to using, such as the prominent use of backward reasoning, guess-and-try, and substitution techniques related to tasks in AO_2 in Danish lower secondary school.

Through collaborative task design and reflection, bi-institutional lesson study thus functioned as a paradidactic infrastructure in which praxeological differences were made explicit not only at the level of what students actually master and where they have their difficulties, but also at the level of teachers' didactic knowledge, including their didactic praxeologies about task design, expectations for how students work with these tasks, and the development of this knowledge thorguh engagement with these praxeological differences.

Across the lesson studies, teachers also observed that designing tasks intended to have students go beyond substitution and guess-and-try techniques – for instance, by aiming for non-integer solutions, as in LS3 (Paper 3) – were insufficient to ensure students engaged in algebraic manipulation. Observations from the research lesson showed that many students continued to rely on arithmetic techniques, such as substituting numerical values or working backwards, rather than engaging in algebraic rewriting.

In practice, these tasks allowed students to remain on the arithmetic side of the cut point described in Chapter 9.1, even though these tasks were designed such that the students could cross the cut point into AO_3.

Through bi-institutional lesson study, teachers thus developed an important form of didactic knowledge: designing tasks that merely *permit* the use of algebraic techniques is insufficient if the aim is to support the development of algebraic praxeologies related to AO_3. Instead, tasks need to be structured so that algebraic manipulation becomes advantageous or unavoidable. This realization represents an initial development in teachers' didactic praxeologies and illustrates how bi-institutional lesson study makes praxeological differences explicit through collaborative task design.

9.3.2 Teachers' observation of students' algebraic work

Across the research lessons described in Papers 3 and 4, teachers observed that only a small number of students were able to engage successfully in algebraic modelling, rewriting, or equation solving as intended. While most students completed the initial arithmetic steps of the tasks, only a handful of the students were able to set up a correct algebraic expression or rewrite it by using algebraic techniques such as the distributive law.

Rather than interpreting these results merely as a lack of algebraic knowledge, teachers began to articulate more precise descriptions of students' algebraic praxeologies. Observations from LS1(Paper 3) and the "think of a number" task (Paper 4) showed that most students could carry out the expected arithmetic steps with numerical values, but considered the task completed once a numerical result had been obtained. Only a few students noticed that the result is "+2" in the "think of a number" task, and even fewer perceived this as something that required explanation (Paper 4).

When variables were introduced, many students experienced difficulties. Teachers described the introduction of the variable *n* as a "barrier" (Paper 4), indicating that students did not treat *n* as a representation of the number in their earlier calculations, but as something new and unfamiliar. Similarly, algebraic expressions were often not accepted as final answers. Several students expressed uncertainty or frustration when no numerical results could be produced and attempted to either substitute numbers or to continue manipulating the expressions without knowing why (Paper 4).

Taken together, these observations led teachers to see students' difficulties as rooted in their way of working with mathematical tasks that focus on calculating numerical results, rather than on treating algebraic expressions as results in their own right. Specific observations supported this. For instance, in LS5 (Paper 3), teachers' expectation that merely having non-integer solutions would discourage substitution, backward transformation, and guess-and-try techniques proved unfounded, as students continued to rely on arithmetic techniques.

The bi-insstitutional setting was crucial for making these interpretations explicit and for enriching them with complementary perspectives. Teachers from lower secondary school contributed detailed knowledge of students' existing arithmetic knowledge and emphasized the distance between what students have been taught and their more result-oriented work and the demands introduced by algebraic modeling and rewriting (Paper 4). Teachers from upper secondary school, by contrast, focused on attention on students' use of algebraic rules, representations, and justifications, for example, in relation to the distributive law and equivalence of expressions. Post-lesson reflection meetings and discussions revealed that these perspectives did not simply mirror institutional differences but enabled teachers to recognize how students' difficulties emerged at the intersection of prior arithmetical techniques and new algebraic expectations.

In addition to these complementary perspectives on students' work, as highlighted in Paper 4 (p. 11), the bi-institutional lesson study also created opportunities for professional learning among the teachers themselves. Teachers from I_2 (upper secondary school) gained first-hand insight into the practices of students and teachers in I_1 (lower secondary school), including the specific challenges students face when working with algebra. This allowed them to better understand the roots of the difficulties observed in earlier stages and to strengthen their focus on the transition problem. Conversely, teachers from I_1 were able to benefit from the input and observations shared by I_2 teachers during planning and reflection meetings, gaining awareness of the needs and challenges that students encounter in upper secondary school, both at the point of transition and later in the curriculum (Paper 4, p. 11).

Although the lessons were jointly planned, post-lesson discussions also revealed uncertainty about how the different parts of the tasks were intended to connect and how students might be supported in moving between them. This was particularly evident in the post-lesson reflection following the "think of a number" lesson (Paper 4), where teachers questioned whether the motivation for algebraic modeling and rewriting had become invisible to many students.

Through these discussions, teachers began to identify the transition from arithmetic to algebra as involving multiple dimensions: changes from numbers to expressions, changes from results to explaining rewritings, changes in language, e.g., the meaning of terms such as "expressions" or "reduce", and in norms for what counts as a completed solution.

In this way, observation of students' algebraic work within bi-institutional lesson study supported the development of teachers' didactic praxis and theoretical knowledge about praxeological differences related to AO_3. It also enabled teachers to develop more precise didactic knowledge of students' arithmetical praxeologies and the specific obstacles involved in the transition to algebraic modeling and rewriting. In particular, teachers came to understand students' difficulties with variables, expressions, and rewriting not as isolated difficulties.

9.3.3 Limits and tensions in bi-institutional lesson study

While bi-institutional lesson study supported the development of didactic and theoretical knowledge, the findings also highlight limitations in its observed potential to develop didactic praxis aimed at addressing the praxeological differences in algebra. Across Papers 3 and 4, teachers from both institutions experienced difficulties in designing tasks that simultaneously necessitated algebraic manipulation, maintained low entry points, and supported students' explanations and justifications of algebraic techniques.

One challenge concerned the use and development of didactic techniques for probing, discussing, and summarizing students' work during the research lesson. Although teachers recognized the importance of such techniques during the post-lesson reflection meetings, they often did not enact them in the research lesson. This difficulty was evident, for example, in LS1 and LS5, where post-lesson discussions revealed that teachers missed concrete techniques for orchestrating collective discussions of students' solutions and for explicitly linking these solutions to the algebraic expression produced by only a small number of students.

These limitations are closely related to the constraints identified in Chapter 9.2. In particular, the absence of a strong paradidactic infrastructure and clearly negotiated roles and responsibilities constrained the extent to which knowledge developed within bi-institutional lesson study could further be used to design research lessons address the praxeological differences related to AO_3. Moreover, institutional constraints, such as curriculum demands and examinations, continued to shape what teachers considered feasible within their usual practice. The lack of a permanent paradidactic infrastructure further limited the possibility for the observed development to continue and extend over time.

An additional obstacle observed in both cases is that the perspectives of the participating teachers may be so different that sustaining a continuously developing dialogue becomes difficult. While this challenge could diminish over time, the case in Paper 4 also shows that bi-institutional lesson study's potentials do not all depend on exchange. Observations that affect teaching in earlier grades, through lower secondary school teachers, or in upper secondary school can remain impactful even if they are not directly shared (Paper 4, p. 11).

Taken together, these findings suggest that the primary contribution of bi-institutional lesson study lies not in an immediate resolution of students' praxeological differences in algebra but more in supporting teachers in developing a sustainable didactic praxis for identifying and addressing such differences, for instance, through task design.

10 Discussion

This chapter discusses how and to what extent this PhD thesis has contributed to the study of its three main subjects: (1) The transition problems in algebra, located between two neighboring institutions, (2) the establishment and conduct of a bi-institutional lesson study aimed at addressing these problems, and (3) the development of teachers' didactic knowledge related to the transition problem in algebra. In particular, we point out limitations in the scope and generality of our results and point out further perspectives and problems for research.

10.1 Subject-specific transition problem

As stated in the Introduction (Chapter 1), Danish students experience the transition from lower secondary to upper secondary school as particularly challenging in mathematics compared with subjects such as Danish and English. This thesis has addressed this challenge by focusing on subject-specific transition problems in algebra and by analyzing them from an institutional and praxeological perspective.

The results presented in Chapter 9.1 show that so-called praxeological differences in algebra exist between the two institutions (Danish lower and upper secondary school). Working with praxeological differences in addressing transition problems has proven to be a useful tool to provide a concrete and well-founded diagnosis of where the transition problem is located.

The results (in Paper 1 and Chapter 9.1) show that the transition problem is primarily related to praxeological differences associated with AO_3, that is, tasks involving the transformation and rewriting of algebraic expressions. Tasks and techniques related to AO_1 and AO_2 are more or less aligned across lower secondary and upper secondary schools, while tasks requiring algebraic transformation or rewriting of algebraic expressions occur to a much lesser extent in lower secondary school, but become central at the beginning of upper secondary school. Students, therefore, leave lower secondary school with techniques that are sufficient for solving first-degree equations through substitution or backward transformation, but insufficient for the algebraic rewriting and transformation expected in upper secondary school.

As also emphasized in Chapter 9.1, in relation to earlier research on the transition from arithmetic to algebra, in particular Filloy and Rojano (1989) notion of a cut, these findings indicate that students are not required to develop techniques related to algebraic rewriting and transformation in lower secondary school, as arithmetic techniques remain sufficient for solving tasks they encounter, such as solving a first-degree equation with substitution. Upon entering upper secondary school, however, students are expected to master further algebraic techniques, for instance, the distributive law, in order to cope with the tasks they encounter early in upper secondary school. The transition problem can therefore be understood as arising from a mismatch between institutional task demands across the two levels. The results obtained, which are examined in detail in Paper 1 and Chapter 9.1, constitute a central contribution to understanding the transition problem in the Danish context and provide concrete and well-founded explanations of where the algebraic difficulties arise.

However, despite the identification of these specific algebra-related differences between the two institutions, it is important to emphasize that transition problems between the two institutions, specifically in mathematics, cannot be explained solely by differences in mathematical content. Mathematics is often perceived as the “difficult” subject, and students’ prior experiences with mathematics, their beliefs about their own abilities, and a general tendency to give up may also play a role in the transition. These aspects are not directly included in this thesis, which is limited to examining transition problems from an institutional and praxeological perspective.

In the work of identifying and addressing transition problems in algebra, I have, based on ATD, defined and applied the concept of praxeological differences as a theoretical and methodological tool. For the reader, the analysis may appear to focus primarily on a “deficit”, that is, on what exists in I_2 but not in I_1. This may raise a legitimate question as to whether there are also elements of algebra in I_1 that do not exist in I_2, and why such differences are not analyzed in a similar manner. The focus on this “deficit” is a deliberate choice, as transition problems arise precisely in the encounter between the praxeologies that students are expected to have acquired in I_1 and the praxeologies that are presupposed in I_2. This focus should therefore not be viewed as a judgment of I_1 as deficient, but rather as an attempt to identify the location of experienced gaps (here termed praxeological differences) in the transition.

Together with the concept of praxeological differences, the analysis and results in Chapter 9.1 focus on the points at which students’ existing knowledge does not align with the expectations when beginning upper secondary school. One consequence of this approach is that other dimensions of students’ transition experiences, such as affective, identity-related, and social dimensions, become less visible, even though these may play an important role in how transitions are experienced by students.

Although these dimensions are not addressed in this PhD project, a possible next step would be to use these praxeological differences as a point of departure for examining how they may also influence other dimensions of students’ transition experiences.

It is also important to emphasize that the two institutions are comparable only to a limited extent. There are contents and practices in both Danish lower and upper secondary schools that do not overlap and therefore cannot be directly related to the transition problems. As presented by Jessen et al. (2017), there are overlaps between the two institutions with respect to content, which is also noted in Chapter 9.1, where tasks related to setting up algebraic expressions and substitution into first-degree equations are identified as types of tasks that exist in both institutions.

Nevertheless, as Jessen et al. (2017) point out, there may be differences in subject cultures (fagkultur in Danish): “Students enter upper secondary school with an approach dominated by everyday applications of arithmetic and plane geometry. In upper secondary school, they encounter demands for a more theoretical and analytical approach and a much higher degree of abstraction” (p. 31).

Thus, even though the two institutions may appear to be more or less aligned on the surface, concrete subject-cultural aspects may influence students’ overall experience of the transition. These aspects lie outside the focus of this thesis.

10.2 Praxis and logos in relation to the PRM and algebra

The praxeological reference model for algebra at the secondary level, developed in Paper 1, as well as the tasks designed in the bi-institutional lesson study, primarily focus on the level of praxis within the framework of ATD. Across lower and upper secondary schools, students work with superficially similar types of task (especially equations) but using very different techniques, so that the praxis blocks are quite different and appear as major transition issues. This focus on praxis was also reflected in the materials from both institutions and formed the development of the praxeological reference model.

By contrast, mathematical reasoning (referring to mathematical rules, definition and so on) is less represented at the logos level in lower secondary school materials. In the ninth-grade final exam, no tasks in the part without aids explicitly address this, and only a small number of tasks in the part with aids require students to explain or justify algebraic rewritings. As shown in Paper 1, only a very small number of students achieved full points on these tasks. This indicates that opportunities to engage with explanation and justification in algebra are scarce in lower secondary school, at least if the ninth-grade exam is a valid measure of what has been learned there.

The tasks analyzed in the development of the praxeological reference model for algebra at the secondary level were primarily situated at the praxis level, although some tasks also included an explicit mathematical logos part. Similarly, the tasks designed for the bi-institutional lesson studies explicitly invited students to explain and justify algebraic rewritings. However, the results from Papers 3 and 4 show that many students experienced difficulties already at the praxis level and therefore did not reach meaningful engagement at the logos level. This suggests that work with logos in algebra is strongly dependent on students' mastery of the praxis level.

From the perspective of ATD, this dependency between praxis and logos is expected. Logos cannot be developed in isolation from praxis; rather, explanations and justifications presuppose a praxis to justify and explain. When students have limited techniques for setting up and rewriting algebraic expressions, the possibilities for articulating explanations and justifications are equally limited. This suggests that didactic techniques aimed at strengthening students' engagement with the logos level cannot be realized in isolation but must be closely connected to the development of students' algebraic praxis.

The praxeological differences identified in relation to praxis and logos have implications that go beyond the transition from lower secondary to upper secondary school. In upper secondary school, students increasingly encounter mathematical activities that require explanation, justification, and generalization, for example, in calculus, proof, and trigonometry. Upper secondary school teachers often say that their students in the first year are often able to identify which trigonometric relation to use in a problem involving arbitrary triangles, such as choosing between sine and cosine relations when given two angles and a corresponding side; but difficulties emerge when students must rewrite the trigonometric relation and isolate a variable, because their limited mastery of algebraic techniques constrains their ability to manipulate these expressions. To the extent this is true, the repercussions of praxeological differences in algebra extend to reasoning and justification later in and potentially throughout upper secondary school, affecting other domains of mathematics than algebra.

These transition problems are not even confined to mathematics alone. Similar challenges can be observed across the natural sciences, where students are required to transform or rewrite formulas to solve problems. For example, in biology, when calculating a person's weight given their BMI (Body Mass Index) and height, students must rearrange the BMI formula $\left(\frac{weight\ in\ kg}{(height\ in\ meters)^2}\right)$ to solve for weight, which requires algebraic techniques. Such algebraic rewritings also appear in chemistry and physics. Even in social sciences, such as economics, students encounter situations where formulas or relations must be manipulated by rewriting or transforming an expression to answer the questions or tasks. These examples illustrate that difficulties at the praxis and logos level in algebra have broad and persistent consequences across subjects in upper secondary school, emphasizing the need to strengthen both algebraic techniques and engagement with the logos.

Finally, the results from the bi-institutional lesson studies illustrate that while work with logos can be made explicit, such work remains constrained when students lack algebraic techniques. Logos can be meaningfully addressed once or while students are developing the techniques for setting up and rewriting algebraic expressions using algebraic techniques. From this perspective, strengthening students' engagement with logos in algebra required sustained attention to the development of praxis, both before and after the transition to upper secondary school.

10.3 Bi-institutional lesson study as an approach

This PhD project has experimented with a format of lesson study that has not previously been explored: bi-institutional lesson study with a focus on transition problems. Papers 2, 3, and 4 together form a series of exploratory cases, all addressing the bi-institutional lesson study. As highlighted in the Papers, these cases provided insight into the establishment of the bi-institutional lesson study, its potential and obstacles, and what teachers gain from participating in such collaboration. By analyzing the six bi-institutional lesson studies, it has been possible to draw concrete conclusions from individual cases while also identifying patterns and generalizable insights across cases.

The results (presented in Chapter 9.2) point to a number of conditions for establishing this format, including the importance of a shared model of the mathematical content and a mutual genuine interest and engagement among teachers from two different institutions. Several obstacles were identified, including the role distributions between teachers from different institutions and differences in teachers' expectations of students' knowledge of algebraic praxeologies.

A shared model of the mathematical content is rarely something teachers possess at the outset of a bi-institutional collaboration. Establishing such a shared understanding is therefore a crucial part of creating a common starting point for planning and teaching. This process is closely connected to kyozaikenkyu, a practice central to Japanese lesson study, although research shows that kyozaikenkyu is absent in around 63% of lesson study formats outside

Japan (Seleznyov, 2018). The difficulties observed in the early phases of the bi-institutional lesson studies can thus be explained through Seleznyov's (2018) finding that international implementations often omit precisely this phase of in-depth engagement with content and curriculum. Fujii (2018) similarly emphasizes that without kyozaikenkyu, lesson planning risks becoming procedural rather than inquiry-oriented.

The need for such a phase early in the establishment is even more important in collaborations involving teachers from different institutions, where differences in curricula and expectations about students' prior knowledge can create obstacles.

By engaging early with curriculum materials from both institutions, though primarily from lower secondary school, teachers were able to negotiate a shared understanding of algebra on three levels: (1) formulation of algebraic models, (2) substitution within these models, and (3) rewriting or operating on algebraic models.

This mirrors the function of kyozaikenkyu in Japanese lesson study, where study of curriculum materials supports the development of shared didactic and mathematical praxeologies (Fujii, 2018; Lewis, 2016).

This process of establishing a shared model not only aligns teachers' understanding of the mathematical content. It also supported task planning and design, anticipation of students' responses, and discussion of didactic techniques, reducing potential misunderstanding arising from institutional differences. In this sense, developing a shared model was essential for the successful establishment of the bi-institutional lesson study format.

The collaboration demonstrated potential in providing teachers with a view of the algebraic praxeologies in the two institutions, as well as insights into their own and each other's didactic praxeologies. A key result was how difficult it was for teachers from both institutions to design tasks for students close to the point of transition, pointing to task design at this level as a central area for further research.

Teachers' difficulties in designing tasks may indicate that their view of algebra and of students' algebraic knowledge in the transition are largely implicit and closely related to their own institutional practices. When these views are made explicit during the bi-institutional lesson studies and translated into tasks, differences in expectations become visible. This highlights task design as an important medium for sharing and developing didactic praxeologies in the bi-institutional lesson study.

A further consideration emerging from the findings is the timing of the bi-institutional lesson studies. Siegler et al. (2012) point out that students' earlier arithmetic skills, particularly their mastery of fractions of fractions in fifth and sixth grade, strongly predict their later success in mathematics, including algebra. Implementing bi-institutional lesson studies in eighth grade rather than ninth grade could therefore provide students with more time to consolidate arithmetic and algebraic techniques and allow teachers to address these praxeological differences earlier. Early intervention may help students build stronger algebraic praxeologies and better prepare them for the transition to upper secondary school.

The results from Chapter 9.3 further suggest that task design plays a crucial role in making praxeological differences visible, but also reveal important limitations. Tasks that were intended by teachers to address algebraic rewriting and transformation (AO_3) did not necessarily function as such in students' work. The findings indicate that teachers (in both institutions) initially overestimated the extent to which students would engage in algebraic manipulation simply because tasks were designed to allow it. Bi-institutional lesson study thus contributed to the development of teachers' didactic knowledge by revealing that addressing the transition problem in algebra requires tasks to be carefully designed so that arithmetical techniques become insufficient, and algebraic rewriting becomes advantageous or unavoidable. This insight connects directly to research on the arithmetic-algebra transition and the notion of cut, and shows how such theoretical ideas are re-encountered and reinterpreted in teachers' collaborative work.

The scope of this project entails clear limitations. A one-year lesson study process is not sufficient to address all aspects of the transition problems. This was, of course, completely foreseeable, so what has been examined here is a more local functioning of bi-institutional lesson study. This limitation reflects Cheng's (2019) argument that lesson study cannot succeed without long-term professional accountability and institutional commitment.

In the Danish context, our project represents an early stage in the development of such a paradidactic infrastructure. In the absence of an established paradidactic infrastructure, teachers themselves had to identify relevant tasks related to algebra. This placed considerable demands on teachers who had limited knowledge of each other's institutions, and the collaboration therefore relied on the teachers' interest and curiosity.

This contrasts with the Japanese context described by Fernandez and Yoshida (2004) and Baba et al. (2018), where lesson study is supported by curricula and teaching materials developed, in part, through careful experience by teachers within established formats like lesson study. Although bi-institutional lesson study does not exist in Japan, if Japanese teachers from different institutions were to collaborate, such transition problems would likely be easier to manage due to the well-established paradidactic infrastructure, which provides support through the Ministry of Education, local authorities, universities, and teacher educators (Paper 4).

A central issue in the bi-institutional lesson study concerns the asymmetry, both in how the lesson studies were conducted and in the teachers' different educational backgrounds. Teachers from both institutions expressed a desire that lower secondary teachers should have the opportunity to observe some research lessons in upper secondary school. The lack of this was a condition for the implementation of the project, for financial reasons, but under different circumstances, research lessons in both institutions could be an important extension of the bi-institutional collaboration.

The results in Chapter 9.2 show that the lower secondary school teacher who delivered the open lesson often assumed responsibility for parts of the planning process that could otherwise have been shared among the teachers. This phenomenon is often observed with novices of lesson study in general.

10.4 Further problems and perspectives

The questions of this project are relevant and important in view of the well-documented transition problems experienced by students in the Danish context, which are expected to continue. It is evident that a one-year bi-institutional collaboration cannot be expected to eliminate these problems even locally, but the project contributes knowledge about how collaboration between teachers from different institutions can be organized locally.

The empirical data consists of the participation and collaboration of a small number of teachers, and it is therefore primarily these teachers who have gained experience with bi-institutional lesson study. To broaden this new approach to addressing transition problems, I have, during my PhD programme, prioritized the dissemination of the results obtained in the PhD project through larger workshops with lower and upper secondary school teachers as participants, as well as through a contribution to a magazine *Gymnasieforskning* (published for Danish high school teachers). This article has also led to invitations from other colleagues, interested in learning more about the results, potential, and the work of this PhD project. This underscores that the project's focus area is not only of interest to me as a researcher but represents a larger societal issue that lower secondary school teachers, upper secondary school teachers, and school leaders are eager to learn more about and address in practice.

Nevertheless, despite these dissemination efforts and the continued interest in bi-institutional lesson study, it appears that it cannot be implemented on a larger scale within the current paradidactic infrastructures, in part because they are separated for lower and upper secondary school. The small scale of this project is therefore not merely a consequence of time or resources. It reflects a more structural challenge: no existing infrastructure easily accommodates bi-institutional lesson study. Lower and upper secondary school teachers in Denmark are working within different institutions and infrastructures, with different careers and professional development paths. This makes it difficult to determine where bi-institutional lesson study could be situated and established within the existing paradidactic infrastructure. The bi-institutional lesson study could only be realized in this project because research funding were available to support it. Scaling up, establishing, and developing bi-institutional lesson study would not only require financial resources and support, but also a new infrastructure where collaboration between neighboring but different institutions is expected and facilitated.

One question that could reasonably be raised, in the Danish case, is whether bi-institutional lesson study in the form considered here is really relevant for all students in a lower secondary school class, when it is known that approximately 70% of the students later continue to upper secondary school. However, transition problems in algebra do not concern only this group. Algebraic modelling and generalization properties constitute central tools in other educational and professional contexts as well, and are therefore relevant for students who follow such pathways.

The choice of lesson study rather than other professional development approaches is justified by the concrete focus on students' transition problems in algebra. Lesson study gives teachers the opportunity to observe students' concrete work over time, thereby providing deeper knowledge into students' work with concrete mathematical content.

Other professional development approaches, such as courses or workshops, have the advantage of reaching a larger number of teachers simultaneously, are easier to scale, and normally more theoretically oriented. However, such approaches would likely not provide the teachers with the same depth of knowledge and work with the mathematical content that gives rise to the transition problems. For example, in the bi-institutional lesson study, teachers can articulate their expectations of students' algebraic praxeologies during planning and reflection meetings and relate these expectations to concrete observations.

The bi-institutional lesson study presented in this project serves as an example of how teachers from neighboring, but different institutions can collaborate on addressing concrete transition problems. While this study focuses on the transition from lower to upper secondary school, similar forms of collaboration may be relevant in other transition contexts, such as the transition from primary to lower secondary school.

This thesis contributes internationally to ATD by defining praxeological differences as a theoretical and methodological tool for analyzing concrete mathematical content that may give rise to transition problems. Furthermore, it introduces bi-institutional lesson study as a new format within lesson study research, using ATD to theorize collaboration across institutions.

Overall, this PhD work and thesis contribute empirical and theoretical knowledge about transition problems in mathematics in a Danish context, as well as methodological and theoretical perspectives on bi-institutional lesson study. These contributions do not lead to definitive solutions, but rather to well-founded hypotheses and points of attention for future research and practice.

11 Conclusion

The PhD project investigates the transition problem in algebra between Danish lower secondary and upper secondary school. To address this aim, the concept of praxeological differences, grounded in the Anthropological Theory of the Didactic (ATD), was developed as a theoretical and methodological tool to identify and analyze differences between the algebraic praxeologies taught and learned in Danish lower secondary school and those expected at the entrance of upper secondary school.

Building on previous research, highlighting teacher professional development as an approach to addressing transition problems, the next phase of the PhD project involved experimenting with a novel form of teacher professional development inspired by Japanese lesson study, namely bi-institutional lesson study. This approach involved participation of teachers from Danish lower and upper secondary schools in all phases of the lesson study, including joint planning, conducting, observation, and reflection on a research lesson. In the present study, the research lessons were conducted in the lower secondary school, by a teacher from there, while the whole group participated in design, observation, and reflection. The study examined the conditions and constraints that appear to facilitate the establishment and process of bi-institutional lesson study as a paradidactic infrastructure, and how this infrastructure can support teachers in developing relevant didactic practices and theoretical knowledge to address the algebra transition problem between the two institutions.

The four research papers that were produced in the PhD study address the different parts of the project: the algebra transition problem, the establishment of bi-institutional lesson study, and the didactic practices and theoretical knowledge developed by teachers across lower and upper secondary school.

To address the first part of the study, the concept of praxeological differences is introduced and developed. The concept is based on a praxeological reference model for algebra at the secondary level, constructed through a praxeological analysis of materials such as textbooks, final exams, and tests from both institutions. Formally, the praxeological differences are expressed as: $\mathrm{MO}^{\mathrm{I}_2} \setminus \mathrm{MO}^{\mathrm{I}_1}$, where $\mathrm{MO}^{\mathrm{I}_2}$ denotes the mathematical organization students are expected to have mastered when entering a new institution I_2, and $\mathrm{MO}^{\mathrm{I}_1}$ denotes the mathematical organization they actually acquired at their previous institution I_1.

In the first part of the project, a praxeological reference model for algebra at the secondary level was developed. At a general level, this analysis showed that secondary-level algebra in Denmark can be described through three algebraic organizations: AO_1 (setting up algebra model), AO_2 (substituting in an algebraic model), and AO_3 (rewriting or operating on an algebraic model).

Based on this reference model, praxeological differences were identified for each algebraic organization. The results show that the algebra transition problem in Denmark is not primarily related to AO_1 or AO_2, but mainly to AO_3. In particular, difficulties arise in tasks involving the rewriting and transformation of algebraic expressions, as well as the solution of first-degree equations using algebraic techniques. Tasks that require students to set up algebraic expressions or to substitute numbers into expressions and equations occur in both lower and upper secondary school to a similar extent. By contrast, tasks belonging to AO_3, such as solving equations like $-3x = 5$ or $3 \cdot (14 + x) = 9$, require rewriting and transformation based on algebraic techniques (e.g., the distributive law) and are more strongly emphasized at the entry to upper secondary school.

At the end of lower secondary school, students are typically expected to solve equations such as $6x + 5 = 41$ or $4 \cdot (x + 1) = 5x$, which can often be solved through substitution by integers, or $4 \cdot (x + 1) =100$, which can be solved by successive "unpacking".

To address these identified praxeological differences, this PhD project subsequently experimented with bi-institutional lesson study as a form of teacher professional development.

The second part of the study examined the conditions and constraints that facilitate the establishment and process of bi-institutional lesson study as a new paradidactic infrastructure. A central contribution here is the empirical investigation of how such a new paradidactic infrastructure can be established with the explicit purpose of addressing transition problems between two connected and neighboring institutions. Based on observations and analyses of the planning and reflection meetings, the results show that the establishment of bi-institutional lesson study requires the development of a shared model of the subject matter constituting the transition problems, in this case, algebra, which can function as a common reference point across institutions. The findings further indicate that the development of such a model requires attention and study of curricular materials (such as textbooks, exams, tests, etc.), as teachers cannot assume shared understandings of algebra or of students' difficulties across institutions.

Teachers' mutual interest and engagement emerged as a central condition for the collaboration, while asymmetric engagement between participants was identified as a potential constraint. In addition, the results indicate that unclear and implicitly negotiated roles and responsibilities function as barriers to both the establishment of collaboration and the collaborative development and sharing of knowledge. In particular, when lower secondary school teachers are implicitly assigned greater responsibility in the planning meetings, the work risks becoming unevenly distributed, thereby undermining the shared nature of the lesson study.

Finally, the findings highlight that the absence of an established paradidactic infrastructure, such as clearly defined roles, facilitators, or knowledgeable others, renders the bi-institutional study fragile and places additional demands on explicit negotiation of roles and responsibilities during its establishment.

The third part of the study examined how bi-institutional lesson study can be used to develop relevant didactic practices and theoretical knowledge aimed at addressing and reducing the algebra transition problem. Based on observations and analyses of planning and reflection meetings from six bi-institutional lesson studies, the findings indicate that this form of teacher collaboration supports the development of teachers' didactic practices, in line with existing research on lesson study. However, the study also shows that the specific potential of bi-institutional lesson study lies in its capacity to address transition problems that are rooted in structurally and institutionally disconnected teaching practices.

More specifically, the study shows that bi-institutional lesson study functions as a productive paradidactic infrastructure through which praxeological differences between lower and upper secondary school, particularly those related to AO_3, are made explicit and become a shared object of inquiry among teachers from both institutions.

Teachers from upper secondary school primarily contribute knowledge about algebraic needs and task designs that are related to the algebraic skills they require, while teachers from lower secondary school contribute knowledge about students' existing arithmetical praxeologies and how these are linked to the needs pointed out by the upper secondary school teachers. These complementary perspectives can be an obstacle in the collaboration, but the potential of bi-institutional lesson study depends on the exchange among the teachers from both institutions.

A central result is that collaborative task design among the teachers plays a key role in making praxeological differences visible. Through the joint design and planning, conducting, and reflection on tasks intended to support students' transition from arithmetic techniques to algebraic rewriting and transformation, teachers developed more precise and shared hypotheses about the conditions under which algebraic techniques emerge in students' work. In particular, the findings show that tasks that merely allow algebraic manipulation are insufficient if the aim is to support the development of algebraic praxeologies related to AO_3. Instead, algebraic manipulation must be made advantageous or unavoidable from the students' perspective, representing an important development in teachers' didactic praxeologies.

Observation and analysis of students' work further supported the development of teachers' explicit (theoretical) and shared knowledge about students' algebraic praxeologies. Teachers came to interpret students' difficulties not as isolated gaps in knowledge, but to some extent rooted in their arithmetical praxeologies. In particular, the transition from operating on numbers to operating on expressions, and from producing results to explain algebraic transformations, emerged as a central challenge in the algebra transition. The bi-institutional lesson study setting was essential in this process, as teachers from lower and upper secondary schools contributed complementary perspectives on students' work. The study shows that meaningful didactic insights may nevertheless arise from observation and reflection, even when they are not fully shared or negotiated across institutions.

The teachers involved in this project work in separate but neighboring institutions, with different educational backgrounds and limited opportunities for collaboration, despite teaching the same students across the transition. Bi-institutional lesson study offers a space for exchange across these transitions, but the study shows that such collaboration cannot be assumed to function independently of the paradidactic infrastructures to support the institutional collaboration. The findings therefore suggest that while bi-institutional lesson study can create an infrastructure for shared knowledge exchange across neighboring institutions, its effectiveness, establishment, and development depend on conditions that enable and support such collaboration.

Overall, the primary contribution of bi-institutional lesson study does not lie in solving students' praxeological differences in algebra. Rather, its contributions lie in supporting the development of a shared didactic praxis and logos that enable teachers to identify and address such differences across institutions.

Taken together, the PhD project contributes (1) theoretically and methodologically by developing the concept of praxeological differences as a tool for analyzing transition problems, (2) empirically by identifying the algebraic rewriting and transformation (AO_3) as the core of the transition problems between lower and upper secondary school in Denmark, (3) methodologically by investigating bi-institutional lesson study as a novel paradidactic infrastructure, and (4) didactically by showing how collaborative task design can support the development of teachers' didactic praxeologies related to algebraic tasks and techniques.

Finally, the aim of this PhD project has been to identify the components of the transition problems in algebra at least as it appears in Denmark and to experiment with a novel form of teacher professional development. By establishing a bi-institutional lesson study, the project has created a shared professional infrastructure in which teachers from lower and upper secondary school can gain experience-based knowledge about each other's practices and engage in dialogue about mutual expectations across institutions. Under ordinary conditions, these teachers rarely have opportunities to observe one another's teaching or engage in professional dialogue about students' learning. This study demonstrates how institutional collaboration can be a significant context to make this both possible and meaningful.

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13 Appendices

13.1 Appendix 1: Template for lesson plan

Template for lesson plan (translated from Bahn, 2018, Appendix F).

Team

School:

Class(es):

Theme:

Lesson study

Title:

Topic:

Intended knowledge:

Research question:

Research lesson

Teacher:

Goals:

Focus points for observation

Background, hypotheses, and questions

Why have we chosen this topic for a lesson study?

Why does it make sense for students' learning to have this lesson now?

Why have we chosen this structure for the lesson?

Why have we chosen the described activities?

Problem:

Lesson plan script

Time	What will happen?	Expected students responses to the didactic milieu (tasks, teacher instruction etc.)	What should the teacher do and say?	Expressions og students'mathematical thinking (actions, formulations)?

Reflection notes

13.2 Appendix 2: Template for lesson report

Template for lesson report (from Østergaard et al., 2020, p. 7; reproduced in full)

The report is a 2-6 page document written to share the ideas and findings developed with other teachers. The main purpose of the lesson report is to present the lesson study team's observations and reflections from practice,

The sections of the report could include:

- Title: (The title of the research lesson)
- Introduction: Previous experiences or ideas related to the theme
- Context: Motivation, goals based on the lesson plan, and hypotheses for the specific activity.
- Lesson plan: See the template above
- Results: Observations from the research lesson (the open lesson) and the main outcomes from the reflection session (can be illustrated with images, e.g., of the board or the students' work)
- Conclusion and future perspectives: Based on the results and external comments and the end of the reflection session.
- References: Resources, articles, websites, etc., used in the lesson study.

Papers

Paper 1

Praxeological Differences in Institutional Transition: the case of school algebra

Abstract. The transition from lower secondary to upper secondary school is a challenging time for many students with algebra as a focal topic. In this paper, we present a new approach to this problem, based on the anthropological theory of the didactic, particularly on what we call praxeological differences between two connected institutions. The methodology involves the construction of a praxeological reference model for school algebra based on documents such as textbooks and evaluation instruments, like national exams and screening tests, from these two institutions. To illustrate this approach, the Danish transition problem in algebra between the lower and upper secondary school is examined as a case study. The results obtained by the students from these evaluation instruments are also a part of the data, to focus on knowledge actually obtained. The results from this case indicate that praxeological difference is chiefly concentrated on rules for rewriting an algebraic model.

Key words. Anthropological Theory of the Didactic, praxeological differences, praxeological reference model, arithmetic and algebra, institutional transition and transition problem

Résumé. Différences praxéologiques dans la transition institutionnelle: le cas de l'algèbre scolaire. La transition du premier au second cycle du secondaire représente un défi pour beaucoup d'élèves, l'algèbre étant un facteur principal. Dans cet article, nous proposons une nouvelle approche à l'analyse de ce problème, fondée sur la théorie anthropologique du didactique, surtout ce que nous allons appeler différences praxéologiques entre deux institutions connexes. La méthodologie implique la construction d'un modèle praxéologique de référence pour l'algèbre scolaire, fondée sur des documents provenant des deux institutions, comme les manuels et les instruments d'évaluation, comme les épreuves nationales et les tests diagnostiques. Afin d'illustrer cette approche, nous examinons le cas de la transition entre le premier et le second cycle de l'école secondaire au Danemark. Les résultats obtenus par les élèves aux évaluations font également part des données utilisées, afin d'examiner les connaissances effectives. Les résultats pour ce cas indiquent que la différence praxéologique est principalement concentrée autour des règles de traitement d'un modèle algébrique.

Mots-clés. Théorie Anthropologique du Didactique, différences praxéologiques, modèle praxéologique de référence, arithmétique et algèbre, transition institutionnelle et problèmes de transition

1. Introduction

The transition from primary to secondary school (usually for students around the age of 12) is widely pointed out as a challenging time for students (Cantley et al.2021). During this time, organizational, developmental, social, and curricular difficulties or discontinuities will disrupt the students' transition and affect their subsequent learning (Cantley et al., 2021).

There is not much research that maps out a specific mathematical domain or theme in the examination of the transition problem from the lower secondary to upper secondary school (Gueudet, 2016), apart from Carraher and Schliemann's (2014) research on early algebra. Gueudet (2016) emphasized that "algebra has long been the "transition topic" par excellence, marking the frontier between elementary and secondary education" (p. 18). In the Danish case, this transition appears mainly between the lower and upper secondary school, as we shall see.

Ruiz-Munzón et al. (2013) point out that "algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge" (p. 4). This crucial role of algebra in the acquisition and understanding of other aspects of mathematics explains the rationale behind our decision to focus on this domain.

Danish students consider the transition from the lower secondary to upper secondary school as particularly difficult in mathematics, compared with subjects like English and Danish, where many students perceive more continuity in content and difficulty (Ebbensgaard et al., 2014).

This study aims to model and map the difficulties of algebra in the transition from the lower secondary to upper secondary school, with the aim to identify the specific mathematical knowledge that contributes most to the perceived differences and difficulties.

We note that in the transition from the lower secondary to upper secondary school, one may find strongly related gaps in arithmetic and algebra since elementary algebra appears at first as a more abstract point of view – or model – of certain arithmetical problems. While this extension from arithmetic to algebra begins already in lower secondary school, algebra is crucial to almost all new subjects in upper secondary school, from basic functions to calculus, analytic geometry and stochastics.

After reviewing previous research on this gap as it occurs internationally, the theoretical framework for the present study, namely the Anthropological Theory of the Didactic and praxeological differences, will be presented. We can then present the research questions of the empirical case of the paper. Subsequently, the methodology for identifying praxeological differences will be presented. Finally, the paper will analyze and shed light on the Danish case.

1.1. The transition from arithmetic to algebra

Research shows that students worldwide experience difficulties in the transition from arithmetic to algebra. For example, Filloy and Rojano (1989) point out that there is a development from arithmetic to algebraic language which relates to the notions and the forms of representation of objects and their operations. In the particular context of solving equations, Herscovics and Linchevski (1994) mention that "the inability to operate spontaneously with or on the unknown indicates the existence of a cognitive gap that can be considered a demarcation between arithmetic and algebra" (p. 63).

Arithmetic and algebra to some extent use the same symbols, but their use of these symbols is different, which can leave students feeling uncertain about their meaning (Kieran, 1990). For instance, in the earlier grades (primary school), students have an operational understanding of the equal sign, meaning they consider the equal sign as a "do something signal" (Kieran, 1981, p. 319); and as emphasized by Welder (2012), a relational understanding of the equal sign, meaning that the equal sign is used to indicate the equivalence of two expressions, is central for learning algebra. For instance, a relational understanding is necessary to manipulate and solve equations, *i.e.*, to understand that the equal sign signifies an equivalence between two expressions is crucial. The students are thus transitioning from understanding the equal sign as a connection between a calculation task and its solution, to understanding the symbol as expressing a symmetric and transitive relation (Kieran, 1990).

By considering the concept of equations, an explanation for this transition problem can appear. Students in primary school have been introduced to and worked with equations in the form $A + B = C$, which means equations where "the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations" (Filloy & Rojano, 1989, p. 19). These are referred to as the "arithmetical" notion of equality in Filloy and Rojano (1989), and methods like numerical substitutions and operating on the numerical terms only are sufficient for solving these equations.

Next, in the transition from primary to lower secondary school, students are introduced to equations like $Ax + B = Cx$, with an unknown on both sides of the equality sign. Students may no longer be able to solve equations using numerical substitutions, but solving this equation now requires operating on the entire equation (Filloy & Rojano, 1989).

In this context, Kieran (1990) points out that:

> The gap that exists between, on the one hand, problems that can be represented by equations with one unknown and that can be solved by arithmetic methods and, on the other hand, problems that are represented by equations with an unknown on each side of the equal sign and that usually must be solved by algebraic methods has been characterized by Filloy and Rojano as a didactic cut (p. 100).

It is, according to Filloy and Rojano (1989), essential to bridge this gap to enable students to transition from an arithmetical mode of functioning to an algebraic one.

Transitions have been studied from different perspectives and theories (De Vleeschouwer, 2010). This paper will examine the transition from lower secondary to upper secondary school (students of age around 16) from an institutional point of view. As De Vleeschouwer (2010, p. 155) pointed out, the transition from one institution to another is not necessarily about the existence of new mathematical content. Rather, this transition, and the problem the students experience in this context, can also be rooted in the fact that the same mathematical content is approached differently in lower secondary and upper secondary school (De Vleeschouwer, 2010). The paper will exemplify this institutional transition problem in algebra from Danish lower secondary school to Danish upper secondary school by using the Anthropological Theory of the Didactic.

2. Theoretical framework and background

The Anthropological Theory of the Didactic (hereafter ATD) introduced by Yves Chevallard (2019), aims to study human knowledge and activity, mathematical or otherwise, as phenomena that are crucially connected to the institutions that aim to develop, facilitate, and constrain them, based on the notion of praxeology (Chevallard, 2019).

According to ATD, praxeology refers to any human practice and activity and consists of two inseparable blocks, *praxis*, and a *logo* block. The praxis block (or know-how) contains one or *more types of task T*, or problems, and *techniques* τ utilized to solve these tasks (Chevallard, 2019). According to Chevallard (2019) the term ‘techniques’ refers to a “way of doing” tasks of type T. With the notation from Chevallard (2019), the praxis block is denoted as follows $\Pi = [T/\tau]$.

From an ATD point of view, no human activity can exist without any description, explanation, and justification. The required discourse on the praxis block is called logos. The logo block consists of two such discourses: a technology θ, namely the discourse utilized to describe, explain, and justify the used techniques, and a theory Θ, which refers to the formal justification of the technology (Chevallard, 2019). With the notation from Chevallard (2019), the logos block is denoted as follows: $\Lambda = [\theta/\Theta]$. The praxis block, $\Pi = [\mathrm{T}/\tau]$, and the logo block, $\Lambda = [\theta/\Theta]$, together form a mathematical praxeological organization (also denoted mathematical organisations or mathematical praxeologies) (Barbé *et al.*, 2005) and is written in the form $\Pi \oplus \Lambda = [\mathrm{T}/\tau] \oplus [\theta/\Theta] = [\mathrm{T}/\tau/\theta/\Theta]$ (Chevallard, 2019).

Mathematical praxeology exhibit varying degrees of complexity: punctual, local, regional, and global ones (Bosch & Gascón, 2006). A mathematical organization (hereafter MO) is punctual if it consists of a single type of task, technique, technology, and theory. When a MO encompasses multiple punctual praxeology that shares the same technology, it is called a local MO. A regional MO comprises several local praxeology that shares the same theory. Finally, a global MO is composed of multiple regional praxeology (Barbé *et al.*, 2005).

We now consider the transition from an institution I_1 to a new institution I_2. I_1 and I_2 are two connected and neighbouring institutions, that is, students pass directly from I_1 to I_2, and depend on what they learned in I_1, at least at the entrance of I_2.

Upon entering the new institution, we assume that students are expected to arrive with a certain minimal mathematical organization $\mathrm{MO}^{\mathrm{I}_2}$. We, furthermore, let $\mathrm{MO}^{\mathrm{I}_1}$ denote elements of $\mathrm{MO}^{\mathrm{I}_2}$ that a certain share of the students have actually learned before leaving the institution I_1. Here, the “certain share” must be fixed and justified according to the context and aims of a given study; it could for instance be the majority of those entering I_2. We then define the praxeological difference (denoted suggestively $\mathrm{MO}^{\mathrm{I}_2} \setminus \mathrm{MO}^{\mathrm{I}_1}$ as all elements of $\mathrm{MO}^{\mathrm{I}_2}$ which are not part of $\mathrm{MO}^{\mathrm{I}_1}$. Notice that these “missing prerequisites” can be entire local praxeology or just minor differences at the level of theoretical discourse, a single technique, etc. Of course, the praxeological difference could also be considered in relation to individual students and their praxeology from I_1 – a decision to include only what a majority failed to learn could reflect a pragmatic and somewhat arbitrary "average" of these individual differences. At any rate, we may often be more interested in identifying central examples than in exactness on items that are, to the expert, not expected to be central. Finally, we note that to find $MO^{I_2} \setminus MO^{I_1}$ we must determine MO^{I_2} first, and this may present greater methodological challenges (a point we return to the methodology).

We hypothesize that to describe transition problems, this concept of praxeological difference has the potential to provide a specific account of praxeological elements that contribute to causing them.

3. School algebra and ATD

Bolea *et al.* (2004) suggest that, in addition to viewing algebra as generalized arithmetic, school algebra should be interpreted as a process of algebraization of previously learned mathematical praxeology, which explains why school algebra is sometimes not treated as a distinct subject in the same way as arithmetic, geometry or statistics (Ruiz-Munzón *et al.*, 2013). Instead, it can be regarded as a general modelling tool of any school mathematical praxeology (Ruiz-Munzón *et al.*, 2013) and one may even choose to "not consider school algebra as a mathematical organization in itself, but as a way of modelling a given mathematical organization" (Bolea *et al.*, 1999, p. 137).

Ruiz-Munzón *et al.* (2013) point out that "algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge" (p. 4), which highlights the essential role of algebra as a tool to address theoretical questions that arise in various domains of school mathematics, such as arithmetic and geometry.

According to Bolea *et al.* (2004), school algebra as a modelling tool has the property of giving "answers to questions related to the scope, reliability and justification of mathematical activity which is carried out in the initial system" (p. 127) and the algebraic model holds the potential to provide a description, generalization and justification of problem-solving processes, while also gather techniques and problems that initially appear unrelated (Bolea *et al.*, 2004, p. 127).

In this paper, we introduce a relatively rough reference model of secondary school algebra which recognizes, on the one hand, that algebraic expressions often arise there as an outcome of modelling processes, but that independent work with algebraic objects is also common, for instance, in solving equations or reducing algebraic expressions that appear without a previous modelling process.

Within ATD, a praxeological reference model (hereafter PRM), is developed by considering local and regional praxeology, as well as sequences of interconnected praxeology (Bosch, 2015). Bosch (2015) notes that the explicit formulation of a PRM for subjects such as elementary algebra can serve diverse purposes. Such a model could, in particular, serve as a crucial tool for the analysis, examination, and description of the algebraic content taught and learned across diverse institutions and can furthermore be used to examine what other elements are missing or can be integrated in any teaching process (Bosch, 2015). According to Barbé *et al.* (2005), among other things, official programs and textbooks may offer "a set of mathematical elements (types of problems, techniques, notions, properties, results, etc.) that constitutes the knowledge to be taught" (p. 240-241). We can view these as elements of an MO, but the level of detail of a PRM depends on the purpose of the model, in particular the questions it is used to investigate.

For our purposes we shall only need a relatively "rough" model, which posits that school algebra at the secondary level consists of three local algebraic organizations (Hereafter AO):

1. AO_1: Set up an algebraic model, based on numerical information (That is, the tasks lead to set up an algebraic expression or equation. Simple example: if a taxi trip costs 7€ per km and there is a start fee of 9€, how can we compute the cost of an arbitrary ride?)
2. AO_2: Substituting in an algebraic model. (Here, the tasks merely involve using given models. For instance, knowing the rule $A = \pi r^2$, what is the area of a circle with radius 7?)
3. AO_3: Rewrite (operate on) an algebraic model. (For instance, knowing that $A = \pi r^2$, how can we compute the radius of circle with a given area?)

These three algebraic praxeology together form a praxeological reference model for school algebra at the secondary level, which can be further detailed (e.g., in terms of techniques or theoretical notion I'd needed. Notice that AO_1, AO_2, and AO_3 are not independent of each other, since they share the same algebraic theoretical discourse, but they do not necessarily build upon each other.

4. Objective of this paper

Gueudet (2008) pointed out that "transition issues can be studied by focusing on mathematical organizations on different levels" (p. 246). This paper examines the transition between lower secondary and upper secondary school by studying the algebraic (praxeological) organizations and praxeological differences between these two institutions, and deals with the following questions:

How can one investigate praxeological differences between two connected institutions through the construction of a common PRM based on documents from these two institutions? In particular, what local algebraic organizations could be relevant to such differences between secondary schools?

More specifically, it has two purposes:

1. To present a general methodology for identifying praxeological differences between two neighboring institutions based on a praxeological reference model.
2. To demonstrate this methodology in action by examining the Danish transition problem in algebra between lower secondary and upper secondary school, while using the previously introduced distinction of three local organizations in school algebra.

5. Methodology

To determine the praxeological difference at the transition between two connected institutions, I_1 and I_2, while focusing on algebra at secondary level, we can use the model introduced above. Concretely the difference can be found as the union of $\mathrm{AO}_1^{\mathrm{I}_2} \setminus \mathrm{AO}_1^{\mathrm{I}_1}$, $\mathrm{AO}_2^{\mathrm{I}_2} \setminus \mathrm{AO}_2^{\mathrm{I}_1}$ and $\mathrm{AO}_3^{\mathrm{I}_2} \setminus \mathrm{AO}_3^{\mathrm{I}_1}$. In other words, we consider the praxeologies of the three main parts of school algebra separately.

At the most basic level, analyzing the algebraic praxeological difference $\mathrm{AO}_\mathrm{n}^{\mathrm{I}_2} \setminus \mathrm{AO}_\mathrm{n}^{\mathrm{I}_1}$ for $n = 1,2,3$ concretely means to identify which algebraic praxis blocks related to AO_1, AO_2 and AO_3 are expected from students in I_2, but according to data from the national exam (see later), they are not learnt in I_1 by a majority of students entering I_2, for instance because they are not assessed at the end of I_1. The analysis of what is expected by the end of I_1 is based on the exam, since the official curriculum is very vague when it comes to concrete mathematical content, and furthermore only has the status of "suggested goals" *(vejledende mål,* in Danish).

To find $\mathrm{AO}_\mathrm{n}^{\mathrm{I}_2} \setminus \mathrm{AO}_\mathrm{n}^{\mathrm{I}_1}$ for $n = 1,2,3$ we begin by determining $\mathrm{AO}_\mathrm{n}^{\mathrm{I}_2}$ for $n = 1,2,3$. The general idea is to do so by analyzing documents such as textbooks and evaluation instruments (like entrance exams and screening tests) used or expected at the entrance of I_2. As mentioned in "Theoretical framework and background", the determination of $\mathrm{AO}_\mathrm{n}^{\mathrm{I}_2}$ for $n = 1,2,3$ may present greater methodological challenges. In the Danish case, this is due to the absence of official requirements as expressed in an entrance test. It is important to highlight that the types of task found at the beginning of textbooks used for the entrance of I_2 may not necessarily reflect expected praxis blocks for students upon entering I_2. These tasks may also indicate what students are supposed to learn after becoming subjects of I_2. The determination of whether solving these tasks is a new learning goal at the beginning of I_2 can be made, in part, by analyzing the level of detail in the examples presented in the textbooks. A careful examination of the specificity and thoroughness with which an example is written or explained can explicitly reveal what students are expected to already know in order to comprehend the example, as well as what new concepts are introduced therein. On the other hand, widely used screening tests at the entrance of I_2 can offer a more extensive and concrete understanding of the expectations at the entrance of I_2.

Ideally and officially, the entry level for upper secondary school corresponds to the exit level of lower secondary school, but in reality, this is not the full truth, as items appearing in review sections or screening tests demonstrate. Thus, considering tasks given to students in the first period of upper secondary school will make it possible to get closer to the actual expectations.

In the Danish context, the first two months of upper secondary school currently involve praxis and theory blocks related to linear functions and models, including algebraic and graphical representations as well as linear regression. During the period from 2017 to 2019, the Danish government required upper secondary schools to assess their students after two months from the start. These tests (a total of 8 tests from STX (The Higher General Examination Programme) called Screening test), primarily focus on linear functions and regression. Algebraic knowledge is required to solve these tasks, right from the entrance of upper secondary school, and therefore they will be used as a main source of indications of the upper secondary school's expectations of students' algebraic knowledge at the entrance of upper secondary school. These screening tests and materials, like textbooks, are password-protected and not accessible to the public. The only publicly accessible screening test is the Silkeborg Screening Test[1].

The identification of $\mathrm{AO}_n^{\mathrm{I}_1}$ for $n = 1,2,3$ is done by analyzing the textbooks and evaluation instruments, used in I_1, and by considering the results obtained by the students from these evaluation instruments. What we look for in $\mathrm{AO}_n^{\mathrm{I}_1}$ for $n = 1,2,3$ depends on what we identified in $\mathrm{AO}_n^{\mathrm{I}_2}$ for $n = 1,2,3$. This will lead to identifying those algebraic praxeologies, related to AO_1, AO_2 or AO_3, which are expected at the entrance of I_2, but are not a part of what students actually learned in I_1. Note here that even though a type of task is present in the evaluation instruments for I_1, it is important to consider how many students actually solve this task correctly. These results will enable a more accurate indication of how many students actually master that type of task. In the Danish context, we analyzed a total of 21 exam sets posed to all students at the end of lower secondary school (9th grade), for the period 2018-2023, and by considering data from the exam results. These exam sets and exam results are password-protected and not accessible to the public. Note that $\mathrm{AO}_n^{\mathrm{I}_1}$ for $n = 1,2,3$ denote the elements of $\mathrm{AO}_n^{\mathrm{I}_2}$ for $n = 1,2,3$ that a certain share of the students has actually learned, and this "certain share" must be fixed, as mentioned in "Theoretical framework and background". In the Danish context, 70% of the students move from lower secondary school to upper secondary school, why it is clear to set "the certain share" to 70%, but it is in reality more difficult to set this fixed, as the prevalence of a type of task should also be taken into consideration, which will be illustrated later in the Danish case.

Note that for the case study, the algebraic praxeological differences will mainly be described at a technical level, as it is easier to access and takes up the most prominence in the written exams, while the theoretical gaps are more difficult to identify (although further studies could usefully attempt to do so).

6. A transition problem in the Danish context: Praxeological differences

6.1. Outline of a more detailed PRM

As mentioned, the praxeological reference model (PRM) for school algebra at secondary level is based on three local algebraic organizations AO_1, AO_2, and AO_3. The concrete PRM in Table 1 – based on our analysis of data as described above - is a slightly more detailed PRM for the Danish case and consists of the three local algebraic organizations, where each of them contains several types of task. Here a distinction is made between three praxeologies of different size and complexity. In building the PRM for the Danish case, we identify a type of task T_i for every algebraic organization and the corresponding technique τ_i used to solve T_i.

[1] https://www.gymnasiet.dk/media/1891/screening_juni15.pdf

AO_1: Set up an algebraic model	AO_2: Substituting in an algebraic model	AO_3: Rewrite (operate on) an algebraic model
$T_{1,1}$: Set up a first-degree equation based on a written description with numerical data. $T_{1,2}$: Set up an algebraic model based on a geometrical situation, usually involving a diagram with symbols attached.	$T_{2,1}$: Substitution of numbers into a linear equation. $T_{2,2}$: Substitution of numbers into a given algebraic expression.	$T_{3,1}$: Rewrite (operate on) a first-degree equation. $T_{3,2}$: Rewrite (operate on) an algebraic expression

Table 1. A praxeological reference model for school algebra at secondary level in Denmark.

AO_1 consists of tasks aimed at constructing an algebraic model and AO_1 is further divided into two different types of task. AO_2 consists of tasks that can be solved by substitution in an algebraic model, both numerically and with letters and variables.

AO_3 involves tasks aimed at rewriting or operating on algebraic models, and it includes a detailed discourse and description of the techniques involved. Based on the praxeological analysis, AO_3 is divided into classes of tasks, including rewriting a first-degree equation and rewriting an expression. Both types of tasks can, for example, make use of a relatively large number of techniques related, for instance, to the commutative and distributive laws, syntactic rules governing the use or non-use of parentheses, or exponent rules. For $T_{3,1}$, certain special techniques involving operations appear in addition to these – like adding some number or expression – carried out on both sides of the equality sign. Such techniques are often used in equation solving but are not used when rewriting an algebraic expression. For that reason, we differentiate between $T_{3,1}$ and $T_{3,2}$ in the PRM. This is a main reason for the distinction of $T_{3,1}$ and $T_{3,2}$in the PRM (Table 1).

An example of a task related to $T_{3,1}$ is:

Solve the first-degree equation: $2(x+1) = 5x - 8$

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$
- τ_2: $+, -, \cdot$ or $\div$ on both side of the equal sign.
- τ_3: Simplify by collecting and reducing similar terms.

An example of a task related to $T_{3,2}$ is:

Rewrite the algebraic expression: $r(5+s)+2rs-2r$

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$
- τ_3: Simplify by collecting and reducing similar terms.

The following sections will outline the $AO_1^{USS} \setminus AO_1^{LSS}$, $AO_2^{USS} \setminus AO_2^{LSS}$ and $AO_3^{USS} \setminus AO_3^{LSS}$ where USS and LSS indicate Danish upper and lower secondary schools, respectively. The overall result will be that the transition problem from Danish lower secondary school to upper secondary school does not have its chief roots in $AO_1^{USS} \setminus AO_1^{LSS}$ and $AO_2^{USS} \setminus AO_2^{LSS}$, since $AO_1^{USS} \setminus AO_1^{LSS} \approx \emptyset$ and $AO_2^{USS} \setminus AO_2^{LSS} \approx \emptyset$, but that the transition problem is concentrated in $AO_3^{USS} \setminus AO_3^{LSS}$.

6.2. The praxeological differences: $AO_1^{USS} \setminus AO_1^{LSS}$

Tasks in $T_{1,1}$ are characterized by the students being given some situation and data, and have to assign some variables (if not given by the task formulation) and set up a model based on the given information. In AO_1^{USS}, these models are linear models, meaning they are first-degree equations or expressions. The techniques used for solving tasks in $T_{1,1}$ enable students to determine which variables are involved, to identify an initial value and a rate of change, and then setting up a linear model in the form of $y = ax + b$ with a as the rate of chance and b as the initial value.

Exercise 2 from STX 2017 (1) Screening test is a task of type $T_{1,1}$ from upper secondary school, where the students must set up a first-degree equation based on a written description. Concretely the task involves setting up a first-degree equation to describe the relationship between the temperature of the water and the time from the start of the measurements, where the initial temperature of the water was 22°C and it increases by 7°C per minute. As mentioned, for solving this type of task, the students have to identify the initial value and rate of change and set up a linear model.

Task of $T_{1,1}$ – and also of $T_{1,2}$ – appear every year in the final exam in Danish lower secondary school for the period 2018-2023, and by considering students' performance in the final exam at lower secondary school, we have that $T_{1,1}$ and $T_{1,2}$ are also contained in AO_1^{LSS}.

Exercise 1 from the ninth-grade exam from May 2023 is an example of $T_{1,1}$ in AO_1^{LSS}. Here, students are required to use the same technique as exercise 2 from STX 2017 (1) Screening test, as they, based on a written description, must determine which variables are involved and then set up a first-degree equation. Concretely, the student is presented with several goods whose prices have increased by 9%. The task requires the student to set up a first-degree equation that can be used to calculate the new price of a product that originally cost x DKK.

30% of the Danish ninth grade students received 2 points, and 22% received 1 points (out of 2 points) for this exercise.

Tasks related to AO_1 occur with the same prevalence in both institutions, as we have observed that the type of task related to AO_1 occurs approximately every second year in screening tests for upper secondary school and in the exam for lower secondary school. So, the prevalence of tasks related to AO_1 is the same in both institutions.

Through an analysis of material from Danish lower secondary and upper secondary school, and by considering students' performance in the final exam at lower secondary school and by considering the prevalence of tasks related to AO_1 for both institutions, it can be concluded that AO_1 occur in both institutions with the essentially same types of task and related techniques. Based on this, we claim that the praxeological difference between lower secondary and upper secondary school is not related to AO_1. In other words, $AO_1^{USS} \setminus AO_1^{LSS} \approx \emptyset$.

6.3. The praxeological differences: $\mathbf{AO_2^{USS} \setminus AO_2^{LSS}}$

As mentioned, the praxeological reference model (PRM) for school algebra at secondary level is AO_2^{USS} involves tasks related to $T_{2,1}$ and $T_{2,2}$. These can be identified in the material from the upper secondary school, and a characteristic task is exercise 16a in Figure 1. Concretely, exercise 16a belongs to $T_{2,1}$ where the technique is to set $x = 5$ and substitute it into the function $y = 2x + 3$.

For y and x, the following relation exists: $y = 2x + 3$ What is the value of y when $x = 5$?

Figure 1. Exercise 16 (Silkeborg Screening test)

It is observed from the final exam in ninth grade in lower secondary school that tasks related to $T_{2,1}$ occur every year for the period 2018-2023. Exercise 7 from the ninth-grade exam from May 2023, which involves solving the following three equations:

- 7.1: $6x + 5 = 41$
- 7.2: $4 \cdot (x + 1) = 5x$
- 7.3: $\frac{x}{2} + 12 = 2x - 3$

This is a characteristic type of task from AO_2^{LSS}. Superficially, it appears to be of type $T_{3,1}$, but in reality – given the techniques the students use – it is not, as we shall now explain.

What characterizes tasks related to $T_{2,1}$ in AO_2^{LSS} is that they have positive coefficients and positive integer solutions from the set $\{1 \dots 10\}$. All the equations that are identified in AO_2^{LSS} have these properties: it is sufficient to use a trial-and-error technique with the solutions in $\{1 \dots 10\}$ and thus get the solution with techniques for $T_{2,1}$, without algebraic operations. The tasks, 7.1, 7.2 and 7.3, are solved correctly by, respectively, 80%, 47% and 29% of the Danish students in final exam at lower secondary school. Based on these observations, we claim that Danish lower secondary school students use a trial-and-error technique with the solutions in $\{1 \dots 10\}$ for solving the tasks 7.1, 7.2 and 7.3. We claim that it is more difficult for the students to use substitution with the solutions in $\{1 \dots 10\}$ in task 7.2 and 7.3, since parentheses and fractions are involved, which could be more difficult to calculate, which is why fewer students can solve task 7.2 and 7.3 correctly. Because if the students had used techniques such as the commutative and distributive laws, syntactic rules governing the use or non-use of parentheses, or exponent rules, the tasks, 7.1, 7.2 and 7.3, would be equally "easy" to solve, since they are all first-degree equations and thus have more or less the same correctness among the students.

Note also that substitution with solutions in $\{1 \ldots 10\}$ is a predominant technique in lower secondary school, even in tasks that on the surface looks like tasks related to $\mathrm{T}_{3,1}$ (e.g. the tasks 7.1, 7.2 and 7.3). Tasks such as tasks 7.1, 7.2 and 7.3 occur every year in the final exam in lower secondary school with the same progression, i.e. where the first task always has a higher correctness among the students and where questions 2 and 3 always involve fractions and parentheses.

Exercise 15.2 from the ninth-grade exam from May 2023 is an example of a task belonging to $\mathrm{T}_{3,1}$ in $\mathrm{AO}_3^{\mathrm{LSS}}$ and it was solved correctly by 35% of the Danish ninth grade students. The task involves determining the area of the base in a pyramid with a rectangular base, given its volume, 40 cm^3, and height, 12 cm. In the task, a sketch of the pyramid is given with a rectangular base, where the base dimensions are 2 cm and 4 cm, and the height from the base to the apex of the pyramid is 9 cm. To find the area of the base, the students have been given the formula $V = \frac{1}{3} \cdot h \cdot G$ where V is the volume of a pyramid, h is the height of the pyramid and G is the area of the pyramid's base. On the surface, the task gives the impression that students need to rewrite the expression and isolating G, but what is characteristic of such tasks in ninth-grade exams is that they all have an integer solution, which is why rewriting does not become a prevailing technique among students, according to the guidance offered to the teachers and the exam results.

Through an analysis of material from Danish lower secondary and upper secondary school, and by considering students' performance in the final exam at lower secondary school, it can be concluded that the same types of task and techniques related to AO_2 occur at Danish lower secondary and upper secondary school. We can therefore conclude that the praxeological difference between lower secondary and upper secondary school is not related to AO_2. Therefore, we conclude that $\mathrm{AO}_2^{\mathrm{USS}} \setminus \mathrm{AO}_2^{\mathrm{LSS}} \approx \emptyset$.

6.4. The praxeological differences: $\mathbf{AO}_3^{\mathbf{USS}} \setminus \mathbf{AO}_3^{\mathbf{LSS}}$

$\mathrm{AO}_3^{\mathrm{USS}}$ involves tasks related to $\mathrm{T}_{3,1}$ and $\mathrm{T}_{3,2}$. These can be found in the material from the upper secondary school, and a characteristic task is exercise 6 from STX 2017 (1) Screening test.

The exercise is about students being presented with an attempt to solve the equation $3x + 2(x+1) + 7 = 5$ based on the following series of rewrites:

$$3x + 2(x+1) + 7 = 5$$

$$3x + 2x + 1 + 7 = 5$$

$$5x + 8 = 5$$

$$5x = 3$$

$$x = \frac{5}{3}$$

and the students are tasked with identifying and describing the mistakes made in these rewrites. Concretely, this exercise belongs to $T_{3,1}$ and tasks related to $T_{3,1}$ in AO_3^{USS} have in common that solving them require the use of techniques where an operation on or with the entire equation is needed.

Notice that the classification of tasks related to either AO_2 or AO_3 is determined by observing what students actually do when they solve an equation. If an equation is solved by using a trial-and-error technique with the solutions in $\{1 \ldots 10\}$ and thus without algebraic operations, it can be characterized as a task in AO_2. However, if techniques involving operation on or with the entire equation are done, then the task can be classified as a task in AO_3. For example, the tasks 7.1, 7.2 and 7.3 from the ninth-grade exam from May 2023 can be classified as either AO_2 or AO_3, but we classify it as a part of AO_2 since it has solutions in $\{1 \ldots 10\}$. For exercise 6 from STX 2017 (1) Screening test , the situation is different.

This exercise illustrates a prevalent type of task, related to $T_{3,1}$, that upper secondary school students are expected to be able to solve at the entrance of upper secondary school.

This task can be solved by the techniques:

- τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$
- τ_2: $+, -, \cdot$ or $\div$ on both side of the equal sign.
- τ_3: Simplify by collecting and reducing similar terms.

From an analysis of textbooks used at the entrance of the upper secondary school, tasks related to $T_{3,1}$ in AO_3^{USS}, are identified as tasks that students should be able to solve at the beginning of upper secondary school.

For example, in an exercise from MAT STX textbook introductory phase, students are tasked with solving the following three equations by hand:

1. $3(14 + x) = 9$
2. $-3 \cdot x = 5$
3. $7 - 2x = 3x - 3$

While these tasks might initially seem like previous ones i.e. tasks 7.1, 7.2 and 7.3 from the ninth-grade exam from May 2023 from lower secondary school, there are notable differences. Students move from lower secondary school, where a trial-and-error technique suffices for solving equations with positive coefficients and positive integer solutions, to upper secondary school, where the techniques (τ_1 and τ_2) to manipulate and operate algebraically become necessary to solve first-degree equations; moreover they can have both negative coefficients, negative integer solutions, and non-integer solutions (as the equations from MAT STX textbook).

Figure 2 shows some tasks, used in the entrance of upper secondary school, which is related to $T_{3,2}$ in AO_3^{USS}.

Simplify the following expressions as much as possible:

1. $\frac{a^4 \cdot b^3}{a^2 \cdot b}$
2. $(a-2b)^2$
3. $(x-1)(x+2)$

Figure 2. Exercise 1, 2, and 3 (Silkeborg Screening test)

Exercise 1 in Figure 2 can be solved by the techniques related multiplication of fractions and exponent rules such as τ_4: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ and τ_5: use the quotient rule $\frac{a^m}{a^n} = a^{m-n}$, while exercise 2 can be solved by the technique about squaring a binomial τ_6: $(a-b)^2 = a^2 + b^2 - 2ab$. Finally, exercise 3 can be solved by the technique τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$.

So AO_3^{USS} consists of types of task related to $T_{3,1}$ and $T_{3,2}$ with corresponding techniques $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ and τ_6.

Very few types of task related to $T_{3,1}$ and $T_{3,2}$ exist in AO_3^{LSS}. We have observed that tasks related to $T_{3,2}$ in AO_3^{LSS} involve tasks where students are not required to perform a rewriting of an algebraic expression themselves, but instead, they need to explain a rewriting of an algebraic expression. Notice that out of 10 final exams with aids (where each exam consists of an average of 20 tasks) for the period 2018-2023, this type of task related to $T_{3,2}$ has occurred in 5 out of 10 final exams as one out of the 20 tasks. Therefore, this type of task occurs to a lesser extent in the final exam for lower secondary school. An example of this type of task is exercise 6.3 from ninth-grade exam from May 2021. The exercise is about students being presented with an attempt to rewrite the expression $n^2 - (n+1) \cdot (n-1)$ based on the following series of rewrite

$$n^2 - (n+1) \cdot (n-1) =$$

$$n^2 - (n^2 - n + n + 1) =$$

$$n^2 - n^2 - n + n + 1 =$$

$$1$$

and the students are tasked with explaining the mistakes made in these rewrites.

By considering students' performance in the final exam at lower secondary school, we shall now examine the extent to which these tasks were solved correctly by students, which is essential to consider in the analysis of matter learnt.

To solve exercise 6.3 from ninth-grade exam from May 2021, where the students' aim is to explain the mistakes that are made in an algebraic rewriting, students need to have acquired the technique τ_1: use the distributive law $a(b+c) = a \cdot b + a \cdot c$. 5% of the students received 3 points, and 15% received 2 points (out of 3 points) for exercise 6.3, which could indicate that although a few tasks of type $T_{3,2}$ exists in AO_3^{LSS}, they can actually only be solved by very few students.

Exercise 6.3, which is a task related to $T_{3,2}$ in AO_3^{LSS}, are correctly solved by a maximum of 20% of the students. This type of low correctness, of a maximum of 35% in general, in the final exam among the Danish lower secondary students is a result that can also be observed in other tasks related to AO_3^{LSS}. It is therefore possible, based on the low student performance in the few and very unambitious exam tasks, to conclude that it is only a small minority that acquires parts of AO_3^{LSS} in lower secondary school.

As mentioned in the previous section, it is possible to observe tasks related to solving a first-degree equation in the final exam for Danish lower secondary school. However, since these equations have solution in $\{1 \ldots 10\}$, we chose to categorize these as tasks belonging to $T_{2,1}$ in AO_3^{LSS}. This gives that tasks which at first sight can be characterized as tasks related to $T_{3,1}$ in AO_3^{LSS}, do not belong to it, which is why $T_{3,1}$ is almost not to be found in AO_3^{LSS}.

In conclusion, AO_3^{USS} consists of tasks related to $T_{3,1}$ and $T_{3,2}$. $T_{3,1}$ contains types of task related to solving a first-degree equation (with negative coefficients, negative integer solutions, and real solutions) by operating on or with the entire equation, while $T_{3,2}$ contains types of tasks related to rewriting and operating on an algebraic expression, which is not limited to linear expressions. On the surface, by observing official tests such as the final exam for ninth grade, we see that in lower secondary school, there are tasks related to solving and operating on first-degree equations, and to rewrite expressions. However, the reality in lower secondary school is that all tasks related to solving first-degree equations can be solved by using a trial-and-error method with the solutions in $\{1 \ldots 10\}$ and thus without algebraic operations. So, in lower secondary school, students can achieve full points by solving a first-degree equation without operating on the equation at all and the problem in lower secondary school is also that tasks related to AO_3 are solved by few students. As we observed, AO_3^{USS} involves numerous rules and techniques, whereas AO_3^{LSS} is almost empty. When examining the very few types of task related to $T_{3,2}$ in AO_3^{LSS}, we noticed that they do not involve students working with expressions, as is the case with $T_{3,2}$ in AO_3^{USS}. Instead, students are only required to explain the simplification of an expression rather than performing the simplification using techniques from $T_{3,2}$. So, based on these considerations, we can conclude that transition problem from Danish lower secondary to upper secondary school is concentrated in to $AO_3^{USS} \setminus AO_3^{LSS}$.

Concretely, we can now say that the transition problem between Danish lower secondary school and upper secondary school lies in the fact that AO_3^{LSS} is almost empty while AO_3^{USS} contains many types of task and corresponding techniques. This means that $AO_3^{USS} \setminus AO_3^{LSS}$ is where the praxeological difference is greatest compared with $AO_1^{USS} \setminus AO_1^{LSS}$ and $AO_2^{USS} \setminus AO_2^{LSS}$. This is thus the reason for the significant algebraic gap and thus the transition problem between these two institutions.

7. Discussion

The present study has aimed to examine transition problems in algebra across institutions. To address the transition problem, our main point in this paper was to present a new theoretical notion *praxeological differences* within ATD, as a promising way to understand and describe a transition problem between two neighbouring institutions. Furthermore, we have presented a general methodology for identifying praxeological differences in algebra between neighbouring institutions, using a praxeological reference model for school algebra. Praxeological differences and the corresponding method can be useful in other institutional transitions as well, such as the transition from primary to lower secondary school, and for other mathematical domains with their respective praxeological reference model. A methodological challenge is that it can be very difficult to identify MO^{I_2}, as there is not always concrete material or tests used at the entrance to I_2. In the present study, this was observed in the Danish case. Another challenge is that it is difficult to assess the knowledge acquired by the lower secondary school students without access to their exam results. The term praxeological difference is a useful concept for use on an individual level, but when considering transition problems, it is the sum of all individuals' actually learned knowledge that is central, which is why access to data such as exam results can be important. A methodical choice we have made in determining the praxeological difference between lower secondary school and upper secondary school, in a Danish context, is to focus on the praxis block. There are two reasons for this. Firstly, we observe that the praxis block, at the technical level, is what creates the biggest challenges for the students. Furthermore, the praxis block has a greater presence in the materials of both institutions, and it is difficult to identify the logos block.

For the Danish case, we have observed that the first-degree equation exists in the material from lower secondary school, but even though they are all characterized by having solutions in $\{1...10\}$ and can be solved by a substitution, we observe that there is also a significant variation in how many students solve the tasks correctly. Exercise 7 from the ninth-grade exam from May 2023 is a task with three different first-degree equations of increasing complexity. The tasks, 7.1, 7.2 and 7.3, are solved correctly by, respectively, 80%, 47% and 29% of the Danish students in the final exam at lower secondary school. The decrease in the number of students who have solved the task correctly may, according to Filloy and Rojano (1989), be because students are used to working with equations in the form $Ax + B = C$, where numerical substitution is sufficient to solve this type of equation. However, Task 7.2and 7.3 from the ninth-grade exam from May 2023 are of the form $Ax + B = Cx$, and according to Filloy and Rojano (1989), students can no longer use numerical substitution for this type of equation. But this is not what we observe in the Danish case. Even equations of the form $Ax + B = Cx$ in Danish lower secondary school have solutions in $\{1...10\}$, so these equations are also solved with a trial-and-error technique. So Danish students solve complicated equations, as termed by Filloy and Rojano (1989), with a trial-and-error technique and substitution, and if they calculate incorrectly during this substitution, they can end up solving the equation incorrectly. Therefore, we claim that Danish lower secondary students do not solve first-degree equations incorrectly because the equations become more complicated, as Filloy and Rojano (1989) point out, since the technique remains the same; however, students may calculate incorrectly, for example, within parentheses or with fractions when using a trial-and-error technique with solutions in $\{1...10\}$.

Based on the concept of praxeological differences and praxeological reference model, we can state that the transition problems in school algebra from Danish lower secondary school to upper secondary school is due to praxeological difference $AO_3^{USS} \setminus AO_3^{LSS}$. According to Kieran (1990), this may be because the transition from an operational understanding to a relational understanding of the equal sign has not succeeded, as mastery of AO_3 requires a relational understanding. As indicated by Filloy & Rojano (1989), we can assert that Danish students complete primary school with an arithmetical notion of equality, which could be the reason why the praxeological difference $AO_3^{USS} \setminus AO_3^{LSS}$ arises.

There are so many techniques in AO_3 that it is probably the most important, compared to AO_1 and AO_2, which contain fewer techniques. We have observed that there are few tasks related to AO_1 and AO_2 in both institutions, and these tasks were solved correctly by a limited number of students in lower secondary school. Consequently, AO_1 and AO_2 do not occupy much space in both institutions. We, therefore, found that the greatest praxeological difference, and where we believe the transition problem lies, is at $AO_3^{USS} \setminus AO_3^{LSS}$.

Transitional problems are therefore not directly caused by the tasks that the fewest students solve correctly in an institution. It is equally about the prevalence of a certain type of task. AO_3 is highly dominant and prominent in upper secondary schools but almost entirely absent in lower secondary school. Consequently, the praxeological difference $AO_3^{USS} \setminus AO_3^{LSS}$ is the largest and, thus, the most important compared to $AO_1^{USS} \setminus AO_1^{LSS}$ and $AO_2^{USS} \setminus AO_2^{LSS}$. Therefore, if the prevalence of a certain type of task is high in I_2 and almost absent in I_1, the praxeological difference $MO^{I_2} \setminus MO^{I_1}$ will be large.

8. Conclusion

The present study contributes to the Anthropological Theory of the Didactic by introducing the concept of *praxeological differences* between two neighboring institutions and presenting a general methodology for identifying these differences based on a praxeological reference model. We assert that praxeological differences, denoted as $MO^{I_2} \setminus MO^{I_1}$, and the corresponding methodology have the potential to address the transition problem between two connected institutions, denoted as I_1 and I_2. We have argued that the praxeological reference model for algebra consists of three local algebraic praxeologies; AO_1: Set up an algebraic model, AO_2: Substituting in an algebraic model and AO_3: Rewrite (operate on) an algebraic model.

Applying this general methodology and the praxeological reference model for algebra, we examine the Danish transition problem in algebra from lower secondary school to upper secondary school by identifying praxeological differences: $AO_1^{USS} \setminus AO_1^{LSS}$, $AO_2^{USS} \setminus AO_2^{LSS}$ and $AO_3^{USS} \setminus AO_3^{LSS}$. Our findings indicate that the transition problem is primarily attributed to the praxeological difference $AO_3^{USS} \setminus AO_3^{LSS}$.

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Paper 2

Establishing a bi-institutional lesson study for the transition from lower secondary school to upper secondary school in algebra

Abstract

Transition problems in algebra between institutions – specifically between lower secondary and upper secondary school – can create a need for collaboration. Such collaboration aims to ease the transition by developing shared knowledge of demands and expectations at both levels. This paper introduces a new approach to this problem: a bi-institutional lesson study. In this approach, teachers from two connected and neighboring institutions collaborate through lesson study to facilitate the transition. The approach is presented within the anthropological theory of the didactic. The paper discusses both the conditions that facilitate the establishment and process of a bi-institutional lesson study, as well as the potential obstacles that may arise. Data from planning and reflection meetings in the bi-institutional lesson study indicate that it is important for teachers from both institutions to share some elements of a common model for the subject matter to be taught. Moreover, genuine interest in each other's practices, combined with a willingness to share and learn from them, appears to support the collaboration. Identified obstacles relate to role distributions that occur during the establishment and process of the lesson study, as well as discrepancies between teachers' expectations and students' actual performance.

Keywords: Algebra, bi-institutional lesson study, institutional transition and transition problems, lesson study, the Anthropological Theory of the Didactic,

1. Introduction

In a study conducted by Ebbensgaard, Jacobsen & Ulriksen (2014) on the transition from Danish lower secondary to upper secondary school, students shared a common experience: while subjects like English and Danish remain relatively consistent in the transition, mathematics poses a greater challenge, where few experience continuity from lower to upper secondary level in mathematics. Such transition problems between two institutions are well known. As mentioned in Cantley, O'Meara, Prendergast, Harbinson & O'Hara (2021), these transition problems can be due to organizational, social, or developmental aspects. These reasons are real, but not always easy or possible to change or overcome. Additionally, Cosan (2024) points out that transition problems can be due to specific subject-related academic differences. According to Cosan (2024), the Danish transition problem in algebra from lower secondary to upper secondary school is not related to setting up an algebraic model or substituting into an algebraic model, but it is to a higher extent related to rewriting or operating on an algebraic model.

This transition problem between two connected institutions can lead to a desire for collaboration between teachers from these two institutions, as identified as a central initiative in addressing such transition problems by Gueudet (2016, p. 18), aiming to ease the transition by developing shared knowledge of demands and expectations at both levels.

Lesson studies developed in Japan and other East Asian countries are used for teacher professional development, where teachers collaboratively plan, conduct, observe, and reflect on a lesson (Stigler & Hiebert, 1999). Lesson study is a well-known teacher development practice used in both lower and upper secondary schools, separately, in countries such as Japan, the USA, and Denmark.

Transition problems and the use of lesson study, separately, are well-known and not new. However, in this article, we propose a new version of lesson study, which is called *bi-institutional lesson study*, conducted across two institutions to facilitate students' transition between them, focusing on a particular area of mathematics (school algebra). The primary aim of this study is to identify the conditions and obstacles involved in establishing such a bi-institutional lesson study to address institutional transition problems. Algebra serves as a case study to explore these conditions and obstacles, as research has identified transition problems in algebra between Danish lower secondary and upper secondary school (Cosan, 2024). Furthermore, algebra plays a crucial role in students' opportunities for further education in fields that involve mathematics.

So, the novel proposal presented in this article is that teacher professional knowledge can be developed across connected and neighboring institutions to facilitate the transition from lower secondary to upper secondary school by using lesson study. This concept of a *bi-institutional lesson study* will be further elaborated on in a later section of this article.

The paper starts with a review of previous international research on lesson study. It then presents the Anthropological Theory of the Didactic as the theoretical framework for the current study, followed by a detailed explanation of the concept of *bi-institutional lesson study.* The research purpose and question are then presented, along with a description of the study's context and methodology. Finally, the paper analyzes and sheds light on the Danish case.

2. Background from research on lesson study and theoretical framework

Lesson study as a means for the professional development of mathematics teachers is conducted by following the so-called lesson study cycle, where a team of teachers works collaboratively to

1. Identify and formulate goals for students' learning and development and plan a research lesson collaboratively aimed at realizing these goals.
2. Conduct the prepared research lesson, with one of the planning members as the teacher, while the other team members observe the lesson and collect data on students' thinking, learning, behavior, etc.
3. Share, reflect, and analyze the collected data from the research lesson to improve and develop the lesson.
4. Refine the lesson based on the discussed and observed data and re-teach the lesson. Finally, write a report that includes a lesson plan.
   (Lewis, 2002).

The research lesson refers to the lesson that is taught in the classroom while the research participants observe it. These participants consist of members of the group of teachers and other external guests.

In Western countries, the typical teacher tends to work in isolation and has individual responsibility for their classes. By contrast, in East Asian countries, by virtue of collaborative efforts such as lesson study, teachers have a more collective approach to their work, having developed a professional language to discuss pedagogical and didactic phenomena (Winsløw, 2011). This approach allows teachers to develop and use didactic knowledge that is shared among a broader group of teachers (Miyakawa & Winsløw, 2019). The four mentioned elements of the lesson study cycle – investigation, planning, research lesson, and reflection – work together to foster changes in teachers' knowledge and beliefs, enhance the professional community, and improve teaching-learning resources (Lewis, Perry & Hurd, 2009).

As pointed out by Lewis, Perry & Hurd (2009), lesson study has the advantage that it "makes various types of knowledge more visible, such as colleagues' ideas about pedagogy and students' mathematical thinking, thereby enabling teachers to encounter new or different ideas, and to refine their knowledge" (p. 286). Throughout the planning phase, the research lesson, and the reflection phase, a written and shared plan, the lesson plan, is central for the teachers' reflections, as it supports teachers' collaborative thinking regarding the teaching and learning of mathematics (Lewis, Perry & Hurd, 2009). Observation of the students in lesson study is a central phase to gain knowledge about the students' thinking, and the lesson plan offers a common reference for all participants in the observation and reflection phase (Lewis, Perry & Hurd, 2009).

As pointed out by Miyakawa and Winsløw (2013), lesson study is not "primarily aimed at revising or refining the didactic practice in the observed lesson, but to develop and strengthen the shared theoretical blocks relating and informing didactic practice in a much wider sense" (p. 204). Lesson study can thus lead to a sharper didactic theory shared by teachers. The more individualistic Western approach to teachers' didactic practice does not include this possibility of developing a common theoretical framework.

In the Danish context, attention has been paid to lesson study within different institutions. In the context of teacher education, Østergaard (2016) used the Anthropological Theory of the Didactic to investigate the possibilities of using lesson study as a format for teacher training to improve pre-service teachers' learning during the internship. In Danish upper secondary schools, lesson study has been used as a method for teacher development, and in Jessen, Bos, Doorman & Winsløw (2023), there is, among other things, a focus on how the teachers have used the theoretical framework of the Theory of Didactic Situation in their lesson study activities. Bahn (2018) investigated what and how mathematics teachers learned about teaching and learning in the classroom through a three-cycle lesson study, analyzed using the Theory of Didactical Situations.

Research thus demonstrates how teachers may develop through collaboration with lesson study within one institution, while we will investigate lesson study across institutions in this paper.

Before moving on, a summary of the theoretical framework, the Anthropological Theory of the Didactic (hereafter ATD) introduced by Yves Chevallard (2019), will be provided to model lesson study. According to ATD, human knowledge and activity can be modelled using the notion of praxeologies. A praxeology consists of two parts, a *praxis* part, and a *logos* part. The praxis refers to the practical knowledge (or the know-how), while the logos part refers to explicit knowledge about the praxis (Miyakawa & Winsløw, 2019). Concretely, the praxis part consists of *types of task T* and *techniques* τ used to solve these tasks. According to Chevallard (2019), the techniques refer to the "way of doing" tasks of type *T*. The logos consists of a *technology* θ and a *theory* Θ. While technology refers to the discourse used to describe, explain, and justify the techniques, the theory is a discourse allowing a more formal justification or clarification of the technology (Chevallard, 2019). According to Chevallard (2019), an institution is a system of actors with different institutional positions and relations to an inventory of praxeologies. These roles may include teachers, students, school managers, etc.

The teaching and learning of mathematics involve the construction and sharing of a mathematical organization (MO), through the construction of a corresponding didactic organization (DO). The didactic organization, or didactic praxeology, refers to the praxis and the logos related to the teaching of a specific mathematical praxeology (Miyakawa & Winsløw, 2019). Mathematics teachers' knowledge thus includes didactic praxeologies, but since the didactic praxeologies naturally take its starting point in a mathematical praxeology, these two praxeologies are inseparable. As pointed out in Miyakawa & Winsløw (2019), the core of mathematics teachers' knowledge is didactic. A central notion within ATD is didactic infrastructure, which refers to the conditions and constraints that affect and shape a teacher's activity in the classroom, as well as the more general aspects of teaching and supervision of students (Miyakawa & Winsløw, 2019). These conditions and constraints, which the infrastructure consists of, depend on a wider institutional system, and while some of these are more generic and apply to all teachers in the respective schools, more specific conditions and constraints related to the teaching of mathematics also exist.

Within ATD, teaching-related praxis that occurs outside the classroom can be described by the notions of paradidactic praxis and paradidactic infrastructure (Miyakawa & Winsløw, 2019). The paradidactic praxis refers to the activities and practices that teachers engage in: these are certainly related to specific teaching tasks, but they are not directly about the act of teaching itself. These activities can include the preparation of a lesson, participation in professional development courses, collaborating with colleagues, etc. The paradidactic infrastructure refers to the conditions and constraints in which paradidactic praxis occurs, and how teachers engage in this paradidactic praxis depends on and is shaped by the paradidactic infrastructure. This infrastructure includes, among other things, institutional frameworks and resources that influence how teachers prepare for and engage in teaching. For example, it may encompass institutional policies on professional development, particularly external support for collaborative development initiatives like lesson study (Miyakawa & Winsløw, 2019).

Lesson study can be described as a study of didactic praxeologies. As mentioned earlier, it involves the tasks of planning the lesson, conducting it, observing it, and reflecting on it. These tasks are not didactic praxis but paradidactic praxis (Winsløw, Bahn & Rasmussen, 2018), and lesson study is therefore labeled as a paradidactic infrastructure according to ATD (Miyakawa & Winsløw, 2019). Furthermore, since a lesson study involves multiple actors - such as a team of teachers, who take on various roles and positions, as well as external guests - lesson study can be considered as an institution within ATD, focusing on certain paradidactic practices.

2.1 A new proposal: Bi-institutional lesson study

In Japan, teacher professional development within a school institution is called *konaikenshu,* which is made up of two Japanese words. The first one *konai* means "in school", and *kenshu* refers to "training", so the term *konaikenshu* refers to "training in school" (Fernandez & Yoshida, 2004).

Fernandez and Yoshida (2004) point out that "what makes konaikenshu unique is that it is a form of in-service professional development that brings together the entire teaching staff of a school to work in a sustained and focused manner on a schoolwide goal that all teachers have agreed is of critical importance to them" (p. 8). Thus, konaikenshu usually occurs within one school institution. Lesson study referred to as *jugyokenkyu*, is a specific and widely used form of *konaikenshu* and typically occurs "one element within a wider "study inside the school"" (Winsløw, 2011, p. 295). Lesson study, therefore, normally takes place within one institution and always aims to develop that specific institution.

In this paper, we propose the concept of *bi-institutional lesson study* as a paradidactic infrastructure, where teachers from two connected but different institutions, called I_1 and I_2, together prepare and conduct a lesson study in I_1, aiming to facilitate the transition between I_1 and I_2. It thus extends the idea of lesson study within one institution.

Bi-institutional lesson study in mathematics allows teachers to investigate what they perceive as problematic in the transition between the two institutions and to develop shared didactic praxeologies. Teachers from I_1 and I_2 get the opportunity to reflect on their own and each other's teaching practices, focusing on and developing shared knowledge about student thinking and learning related to somewhat problematic mathematical praxeologies within the transition. Unlike usual lesson study, bi-institutional lesson study involves teachers from both institutions collaborating to conduct a lesson study in I_1. To support this collaboration, familiarity with the curricula of both institutions is important, although a detailed study of I_1's curriculum, including textbooks and exams, is particularly important.

A bi-institutional lesson study focuses on serious discontinuities or transition problems between two institutions. For example, it may be relevant to implement it in educational transitions such as the transition between primary school and lower secondary school, lower secondary to upper secondary school, upper secondary school and university, etc.

As with lesson study, a bi-institutional lesson study consists of a group of teachers, but here the teachers are from two different institutions. In terms of ATD, these teachers can take on various roles and positions within the lesson study. There are no fixed guidelines for which role the teachers from institutions I_1 and I_2 are expected to assume (see the 'Context' section for a description of how this is exemplified in the Danish case).

The external commentator, sometimes called the knowledgeable other, has the task, at the end of the reflection session, to make comments with, according to Fernandez, Yoshida, Chokshi, & Cannon (2001, p. 26), three goals: "(1) to provide a different perspective on the lesson study work of the group; (2) to provide information about the subject matter content, new ideas, or reforms; and (3) to share the work of other lesson study groups" (p. 26).

The role can also be assumed by the facilitator or a member of the teacher team, as long as they can meet the three goals for the knowledgeable other. There may, thus, be different distributions of roles and positions in a bi-institutional lesson study, as is the case for a lesson study, which is why a bi-institutional lesson study is to be considered as an institution set up to study and develop didactic praxeologies. What sets a lesson study apart from a bi-institutional lesson study is that the latter consists of teachers from two different institutions, who may have different mathematical and didactic praxeologies, as is the case in the Danish case described in the 'Context' section. With bi-institutional lesson study, the teachers from both institutions have, as a starting point, the aim of developing a common didactic praxeology. If the development of mathematical praxeologies occurs, especially for the teachers from I_1, this could be an additional benefit.

3. Research purpose and research questions

This paper will examine the following questions:

What conditions appear to facilitate the establishment and process of bi-institutional lesson study, and what obstacles can occur?

As a result, we aim to identify conditions, constraints, and obstacles for establishing a bi-institutional lesson study, observed in various critical moments of the study. This will be examined in a Danish context, specifically for the transition between lower secondary and upper secondary school in algebra, as previous research shows the existence of certain problematic *praxeological differences,* referring to differences between mathematical knowledge students actually acquire at lower secondary and what they are expected to know at upper secondary school in this particular area of mathematics (Cosan, 2024). Studying this transition problem is a common interest for Danish lower secondary and upper secondary schools, as the former have the desire to prepare students for upper secondary school, while the upper secondary school teachers have an interest in gaining greater insight into lower secondary school and receiving better-prepared students.

4. Context

This section presents the context of the study, how the data collection occurred, and how this data has been analyzed.

In the Danish context, teachers from lower secondary school (I_1) and upper secondary school (I_2) differ in both their initial training and mathematical knowledge. Teachers from upper secondary school, I_2, are expected to have a stronger background in the subject matter, as they have a university degree focusing on the mathematical content, but with little emphasis on the pedagogy. On the other hand, teachers from lower secondary school (I_1) studied at a university college, where the focus is more on the didactics and pedagogy, and generally possess a deeper knowledge of these areas. This difference in academic background is important in understanding the roles in this bi-institutional lesson study, as teachers from I_2 generally possess deeper subject matter knowledge.

To illustrate how a bi-institutional lesson study unfolds in practice, this section presents a specific Danish case from the school year 2023-2024. It involves six lower secondary teachers - James, Alice, Susan, Max, Charlotte, and Laura - and two upper secondary school teachers, Henry and Emily, from the same municipality, who engage in bi-institutional lesson study. All names are pseudo-names. The teachers were divided into two groups, each comprising three lower secondary school teachers and one upper secondary school teacher. The upper secondary school teachers had previously participated in the TIME (Teachers' Inquiry in Mathematics Education) project. This project aimed to investigate how a team of teachers can collectively improve their practices through ordinary lesson study. Through this project, the upper secondary school teachers gained experience in using lesson study in upper secondary school (Jessen, Bos, Doorman & Winsløw, 2023).

As a starting point, the teacher from I_2 would serve as the leader of the teacher group, due to experience with lesson study. While teachers from I_2 may start as leaders, they could transition to a more facilitative role as teachers from I_1 gain experience in lesson study. Both teachers from I_1 and I_2 can evolve to become facilitators, but the leader's primary responsibility remains overseeing the process. A teacher from I_1 could also take on this task. The key is to articulate these roles from the start.

The teachers participating in the present project initially attended a workshop introducing lesson studies and related research. They were then introduced to the project's purpose and plan. Presentations on lesson studies in lower and upper secondary schools were given by researchers with experience in lesson study at these levels. Time was allocated for brainstorming and discussing algebra lesson problems related to the transition problem. Finally, teachers received a handbook on the practicalities of lesson study, and a lesson plan template (from Appendix F in Bahn (2018)).

Each group has planned and conducted three lesson studies, resulting in a total of six lesson study cycles carried out with lower secondary school students in their final year (ninth grade), shortly before their transition to upper secondary school. Each lesson study was planned and conducted over 4-5 weeks and followed the model outlined in Fig. 1.

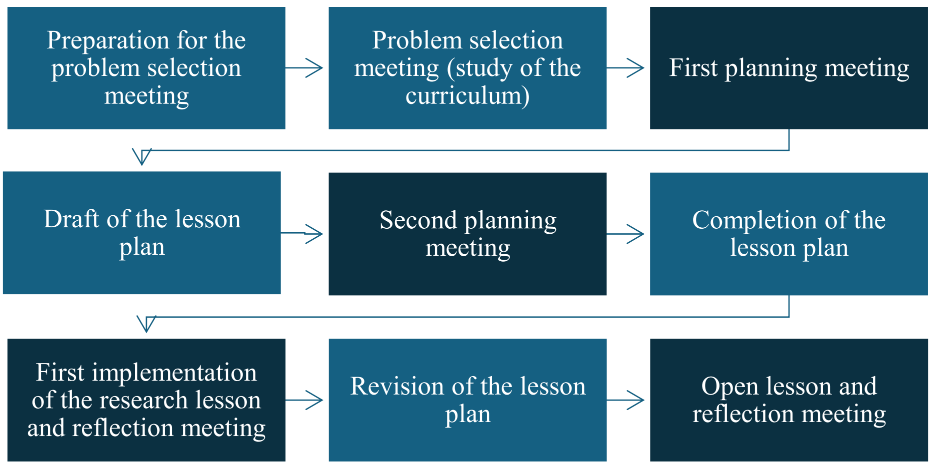

Fig. 1 Structure of one bi-institutional lesson study.

Here is how the groups' work was organized:

Before the problem selection meeting, the upper secondary school teacher prepares suggestions for algebra tasks relevant to the transition problem. At the problem selection meeting, with one lower secondary and one upper secondary school teacher participating, a study of the curriculum in lower secondary school happened, followed by a discussion of how the suggested tasks align with it. The two teachers then selected an algebra problem inspired by the final exams of lower secondary school and entry-level tasks worked on in upper secondary school. They agreed on a lesson goal and a problem for lower secondary school students, focusing on constructing an algebraic model or on transforming such a model (cf. Cosan, 2024)

All four teachers participated in the first and second planning meetings to identify student learning goals and draft a lesson plan. Between these meetings, the upper secondary school teacher and one lower secondary school teacher completed parts of the lesson plan, including the arguments behind the problem chosen and the learning goals of the lesson.

Next, the first lesson realization and reflection meeting involved the entire lesson study group and, sometimes, an external commentator. Revisions to the lesson plan could occur before the final open lesson, taught by the lower secondary school teacher who attended the problem selection meeting. Both lesson study groups, an external commentator, and a reflection session leader participated in the open lesson. The cycle ended with drafting a short practice report (cf. Miyakawa et al., 2019).

The upper secondary school teachers acted as external consultants in relation to the problems identified as central to the transition. These teachers had lesson study experience. The lower secondary school teachers attended both planning meetings, but one of them (the one who taught the open lesson) also participated in the problem selection meeting. All three lower secondary school teachers had the opportunity to teach the first implementation of the research lesson, which did happen, but the teacher responsible for the open lesson (in the case analyzed, this is Alice), primarily also taught the first implementation. The decision regarding who will teach the research lesson is made before the lesson study cycle starts.

The author's role was to attend and facilitate planning meetings, ensure lesson plan completion, and address further practical and theoretical questions related to lesson study. These included logistical questions and inquiries about lesson study and how it is conducted in Japan, which the author has attempted to answer based on existing literature, research, and personal experience in Japan. While the author facilitates the planning meetings, another facilitator (in the case analyzed, this is Jack), with experience in lesson study, facilitates the reflection meetings.

5. Methodology

As the main interest here is the conditions, constraints, and obstacles affecting the teachers' paradidactic praxeologies – planning, observation, and reflection – the analysis primarily focuses on data from planning and reflection meetings, which were audio recordings that were subsequently transcribed (in the original language, Danish).

In the transcriptions, the author has focused on episodes where the development of shared theory blocks seems to *advance in critical ways,* such as building common justifications of didactic praxis through sharing and combining knowledge from the two institutions. Conversely, the author also examined episodes where the teachers seem to *run into critical obstacles,* for example, due to different norms or assumptions related to the two institutions.

Concerning mathematical praxeologies, the author specifically looked for situations where the teachers discussed their understanding of algebraic concepts and tasks. The author focused on moments in the transcriptions where different interpretations or misunderstandings arose among the teachers due to a lack of a common model for algebra. These moments were marked in the data analysis process, enabling us to identify key points where the teachers actively worked to establish a common understanding of algebra.

Similarly, the author strived to identify critical moments related to didactic praxeologies, where the teachers developed new shared knowledge about teaching through the planning meetings and reflection meetings, and through the confrontation of observed student performances and prior expectations. The latter could be identified in moments where discrepancies arose between the teachers' expectations of students' performances and the actual student behavior, resulting in reflection on possible adjustment of teaching.

The analysis of critical moments constitutes key sources in our examination of how bi-institutional lesson study is facilitated and developed. These moments are crucial because they mark moments when teachers are confronted with differences in their mathematical or didactic theory blocks related to their institutions, and where they develop new mathematical or didactic knowledge. With these moments, it is also possible to observe the teachers' expectations of students' performances and actual student behavior.

By a careful analysis of such moments, the author can identify and exemplify key potentials and difficulties for teachers to establish shared knowledge about the mathematical and didactic practices that the bi-institutional lesson study is set up to investigate and develop.

6. Analysis and results

Before presenting the observed conditions, constraints, and obstacles for the establishment and process of a bi-institutional lesson study, the following section will briefly describe the planned research lesson and the selected task. As we are interested in the establishment of a bi-institutional lesson study, the following section focuses on data from the first lesson study of the group, consisting of the upper secondary school teacher, Henry, and the three lower secondary school teachers: Alice, Max, and Laura. This group was chosen because the planned lesson and task were representative of those used in the other group, and the structure of the data collection was identical across groups. Therefore, the observations presented here provide insights that are applicable to both groups.

6.1 Lesson: The area of the hexagon

In the first bi-institutional lesson study, during the planning meetings, the teachers agreed on which task the research lesson should be based on and how this lesson should be structured. The following will briefly describe the planned lesson and the selected task.

The designed task is:

What is the area of the hexagon?

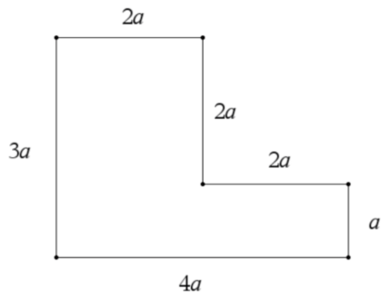

Fig. 2 The task for the research lesson

During the planning meetings, the teachers agreed that the objective of the task is for the students to set up an expression for the area of the hexagon and to understand that there can be different correct expressions for the area that are equivalent through rewriting. Furthermore, with this task, the teachers want the students to be convinced that $a^2 = a \cdot a$ and $a^2 \neq 2a$. These goals for the lesson are also explicitly stated in the lesson plan. At the planning meetings, the teachers identified five different expressions for the area of the hexagon, all emphasizing that $a^2 = a \cdot a$ (see Fig. 3). These five expressions represent those actually proposed by the teachers; other formulations exist, but they were not suggested during the meetings and are therefore not included. In the validation of the solutions of the task, the teachers should emphasize the distributive property to explain why different expressions are equivalent. These are explicitly written in the lesson plan.

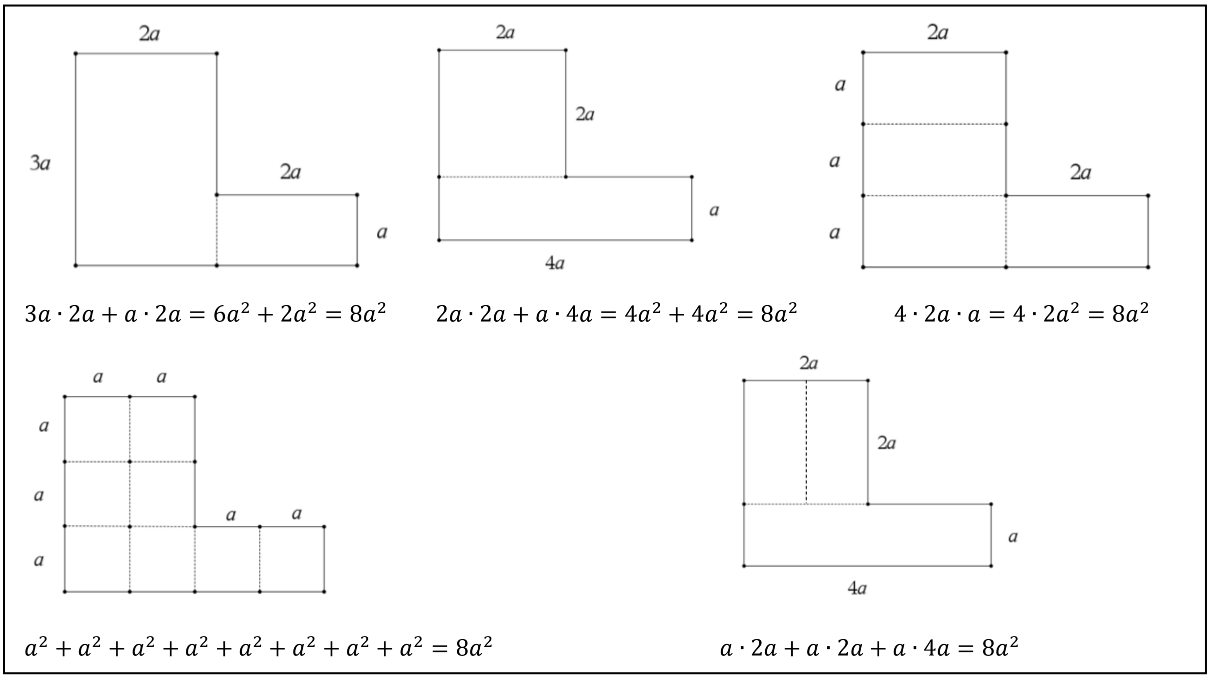

Fig. 3 Five different expressions for the area of the hexagon

6.2 The Need for a Common Model in Algebra

An important condition for the establishment of a bi-institutional lesson study is that teachers from both institutions have some elements of a common model for the subject matter to be taught. This became clear through several episodes during the meetings, where the introduction of a rough distinction of the main kinds of tasks in school algebra was introduced, and subsequently led to more fruitful discussions. The common model described below does not imply a single way of understanding algebra. It reflects how the teachers in this collaboration collectively frame algebraic tasks within their institutions, based on their study of the curriculum and expectations across the two institutions, with reference to research (Cosan, 2024), presented by the author during the meetings. In the following, it is exemplified how teachers actively work to create a common model of algebra through concrete actions and discussions in the planning and reflection meetings.

Before the first problem selection meeting, Henry prepares suggestions for algebra tasks that could be relevant to take as a starting point for the transition between the two institutions. At that time, teachers from both institutions had not had the opportunity to discuss possible tasks related to algebra. In preparation for the first problem selection meeting, Henry examined algebra tasks from the ninth-grade exam for lower secondary school. During the meeting, Alice and Henry studied the curriculum in lower secondary school related to algebra, followed by Henry presenting four examples of tasks related to algebra: three from the exam, and one task he used at the entrance of the upper secondary school.

During the first planning meeting, these ideas for tasks for the research lesson were discussed, mainly with input from Henry. Specifically, types of task related to setting up and rewriting expressions, solving first-degree equations, relationships between two geometric figures with a focus on scale, representation forms, and linear models were identified and discussed by Alice and Henry. After this discussion, Henry summarized by saying: *"What actually bothers them* *is solving equations and mathematical expressions",* referring to the students' difficulties with solving equations and mathematical expressions.

During the planning meeting, the facilitator (the author) identified the need to present a preliminary analysis of the transition problem from Danish lower secondary to upper secondary school in algebra, based on Cosan (2024). This was because the teachers often discussed that the students had difficulty solving equations. However, since equations can be solved both by guessing and substitution or algebraic manipulation, and different difficulties may exist for each types of task, the facilitator saw a need to distinguish between these types of task.

The facilitator does not present the full model (from Cosan (2024), but merely describes the basic kinds of school algebraic tasks as: 1) Setting up an algebraic model, 2) Substituting in an algebraic model, and 3) Rewriting or operating on an algebraic model. This model provided a common starting point for the teachers. It can thus be beneficial in the establishment of a bi-institutional lesson study to have a preliminary analysis of the common transition problem between the two institutions and a successful communication of this to the teachers, for example, in a common model. This distinction guided the teachers in selecting a task for the research lesson that addressed the transition problem between the two institutions.

With this common model of algebra, Henry notes that students often find it difficult to solve equations by rewriting with opposite operations. In the planning meetings, he mentioned that students who have transitioned from lower secondary school are mostly able to solve equations through guessing and substitution, but they often struggle to use the rewriting technique with opposite operations. While discussing what could be interesting and relevant as the starting point for the research lesson, the facilitator observed that both teachers agreed that setting up an algebraic model and operating on it is something the students find difficult.

The facilitator (the author): What you have in common and where you agree must therefore be the set up and rewriting of expressions and the solving of equations.

This quotation demonstrates that the teachers not only identified the difficulties they believe students encounter but also agreed on what the goal for the research lesson should be, based on these identified difficulties. This shared understanding of algebra and the identification of the research lesson goal enriched the teachers' discussions, making them more concrete and constructive. For example, during the reflection meeting, one teacher provided critical observations and suggested improvements. Specifically, it was observed that none of the students wrote $a \cdot a = a^2$ despite the lesson plan foreseeing that this is presented. Based on this, one of the teachers says:

Max: I think one of the reasons why no one ends up including a^2 is that you (Alice) do not simplify the expression. You do not multiply the a's together. I think that could be it. It is simply because you do not start by multiplying the a's together, so there was not a single instance where there was anything, and that is why they do not go any further

Max's comment illustrates the importance of having a common model for algebra. When teachers share a common model for algebra, it becomes easier to identify and discuss specific mathematical praxeologies and related didactic praxeologies. In this case, Max points out a particular difficulty in students' understanding of algebraic expressions, which, according to him, may result from insufficient explanation in Alice's teaching of the research lesson, despite the lesson plan foreseeing that this is presented.

This shared model of algebra is useful in the reflection meeting, as it allows the teachers to know exactly what to observe in the research lesson and simultaneously enables a structured discussion among the observers. This condition for a shared model may also allow the teachers to develop shared didactic praxeologies. Concrete statements, for example, from the reflection meeting where there was a discussion about why few students could set up an expression, and possible reasons were considered, show the benefit of a common model. For example, Alice had expected that students could easily set up expressions and recognize that $a \cdot a = a^2$ as she mentioned in the reflection meeting: *We expected that they would be able to set up an expression with reasonable confidence.*

We cannot know how the reflection meetings would unfold without this common model for algebra. However, we assert that with this common model, the discussions and reflections become concrete and productive. The outcome of these observations is the hypothesis that having a rough common model for school algebra is potentially an important condition for the success of both planning meetings and the reflection meeting.

6.3 Interest and engagement in a bi-institutional lesson study

To successfully establish a bi-institutional lesson study, teachers from both institutions should have a genuine interest in each other's practices and be willing to share and learn from them. This was evident in several instances where the teachers' engagement and interest, as well as the lack thereof, played a crucial role. During a planning meeting, Henry demonstrated a keen interest in the lower secondary school's didactic practices by asking about their methods for teaching algebra to eighth or ninth grade and expressing a desire to build upon their ideas:

Henry: I am a bit curious about how you usually teach equation solving in eighth or ninth grade. How do you do it? ... The reason I am asking is that I think it would be great to take something you have done and twist it.

Henry's remark illustrates important aspects of the collaboration between teachers from different institutions. By using the phrase "*I am a bit curious*", he demonstrates his genuine interest and positions himself as eager to understand and learn from the lower secondary school teachers. By expressing interest in this manner, Henry acknowledges that the teachers from the lower secondary school are the experts on their students' needs and difficulties. This approach helps to establish a respectful dialogue where Henry openly admits his unfamiliarity with the lower secondary school and shows that he is there to gain insight from their experiences. Henry's remark: *"The reason I am asking is that I think it would be great to take something you have done and twist it"* also shows his motivation to improve existing didactic practices from the lower secondary school, rather than rejecting or replicating them, indicating his respect for their teaching practice and the value in their practices which is crucial for a successful collaboration between teachers from two different institutions.

The observation suggests that this interest and engagement can lead upper secondary school teachers to gain new knowledge on didactic and mathematical praxeologies concerning algebra present in lower secondary school.

The same interest in upper secondary school practices is less pronounced among the teachers from lower secondary school. Since the bi-institutional lesson study focuses on lower secondary school practices, we hypothesize that the lack of such interest from lower secondary school teachers about the upper secondary school's practices is due to their focus on the specific lesson and what is directly relevant and current for the lower secondary school. For example, during a discussion in the planning meetings, the lower secondary school teachers were primarily concerned with practical aspects of the current research lesson, as seen in the following statements:

Alice: We have a blackboard that can actually be divided into two, and we also have a whiteboard. We also talked last time about introducing the idea of multiple expressions. If there is already an interactive board that is divided into two, and a whiteboard that is divided into two, it should guide students towards the idea that… It is sort of a nudge, to nudge them towards the idea that there are different expressions.

Max: What we hope the students will understand afterwards is that just because an algebraic expression is written differently, it can still be the same.

These episodes demonstrate that lower secondary school teachers' are focused on how to best structure the research lesson to guide their students toward the intended learning outcomes. For example, their discussion about the use of different boards can be understood as an effort to nudge students towards recognizing equivalent algebraic expressions. However, this greater focus on their own didactic practices may reduce the opportunity to explore how the didactic practices from upper secondary school could inform or inspire them in the current research lesson. This is illustrated in other episodes where the upper secondary school teacher mentioned their practices without any follow-up questions or interest from lower secondary school teachers, as in this situation:

Henry: We have so many balls in the air. Cool! I would say that in upper secondary school, something like this [pointing at task related to representation forms] is difficult for them (in the first year), but it does not bother them that much. What bothers them is solving equations and mathematical expressions. They keep encountering that in math.

Alice: [pointing at task related to representation forms] They [refers to her own students] can often work their way out of something like this by drawing.

In these episodes, we see Henry sharing detailed experiences about the challenges faced by upper secondary school students, especially with algebraic expressions and solving equations. His comments allow lower secondary school teachers to gain knowledge about the difficulties that upper secondary school students encounter. However, neither Alice nor Max follows up with questions to learn more about didactic practices in upper secondary school. Instead, Alice focuses on how her students would approach the task. This lack of engagement reflects a limited interest from lower secondary school teachers in understanding how teaching practices work in the early stages of upper secondary school. This can diminish the possibility for teachers to gain a deeper understanding of the collaborating institution, which may be an obstacle for the establishment of a successful bi-institutional lesson study due to the asymmetry between the two institutions.

6.4 Different roles in a bi-institutional lesson study

One important aspect, which is addressed in detail below, that can both facilitate and hinder the collaboration between teachers from different institutions in a bi-institutional lesson study is the roles that arise during the planning and reflection meetings. These roles are seen as both a condition and an obstacle for the establishment of a bi-institutional lesson study.

During the planning meetings, it was observed that the teachers discuss and plan the lesson with a collaborative approach, where everyone contributes with ideas and suggestions. Despite the research lesson being considered a collective product, responsibility is often placed on Alice, who taught the lesson, creating potential obstacles for the entire bi-institutional lesson study. Another observation during the planning meeting is that the other teachers, both the upper secondary school teacher and the lower secondary school teachers, direct questions to Alice about how she, for example, had planned for the students to share their solutions on the board. Such moments, which can indicate placing responsibility on Alice, can be an obstacle for the entire bi-institutional lesson study, as Alice might end up feeling that she is handling the whole research lesson alone. It is therefore important in a bi-institutional lesson study, as in any lesson study, that the planning of the research lesson and lesson plan is a shared responsibility. Imbalanced responsibility can hinder collective decision-making and the development of shared didactic knowledge among all teachers, which can affect the establishment of a collaborative environment. During the planning meetings, a conversation took place about how the students should share their solutions to the task on the board:

Henry: But you also have a plan, right?

Alice: Yes (laughs a bit), but at least it is something regarding the lesson plan when it needs to be handed over. It would be nice if it was clearly stated.

Henry: Do you plan to distribute it on a large sheet?

Alice: I am thinking, have we written it down somewhere that each student gets something like this?

As it is illustrated in the episode, Henry directly asks Alice whether she has a plan and implicitly places the responsibility on Alice. On the other hand, Alice responded: "*have we written it down somewhere?*", implying that this belongs to what should be written in the lesson plan. Alice indicates with her comment that it is a collective decision and not something she should be solely responsible for. Her use of *"we*" contradicts Henry's *"you"* and although Alice tries to emphasize that the lesson is a shared responsibility, the conversations still reveal an underlying assumption that the responsibility for the lesson is assigned to Alice. During the planning meetings, all teachers plan the lesson together, but in case of questions or confusion, it is observed that Alice is held accountable for the lesson.

In the planning meetings, we also see that Alice is usually busy filling out the lesson plan instead of participating in the discussions, which is where the teachers potentially develop didactic and/or mathematical knowledge. This assignment of responsibility, which Alice has perhaps unconsciously been taking upon herself, seems to hinder her engagement in the discussions and instead focuses on the research lesson planning.

For example, it is observed that while Max and Henry discuss possible student responses (Fig. 3), Alice is not involved in this discussion at all. This lasts for about 10 minutes. After this conversation, Alice says, *"I will just write them down,"* referring to her intention to write down the five possible student responses (Fig. 3) in the lesson plan.

The partially conflicting expectations in roles become even clearer in the first reflection meeting after the first trial. During this meeting, reflections and discussions mainly occur between Alice and the facilitator. These discussions mainly revolve around Alice's curiosity about what she can do differently to make the lesson successful and ensure the students achieve the desired learning objectives. Thus, collective thinking is not prominent, and it appears that Alice is making decisions on her own.

These roles mentioned earlier, which emerge in the planning of the research lesson and the decisions regarding the lesson content (whether it is a collective responsibility or primarily Alice's), can both be necessary conditions and obstacles to the establishment of an effective bi-institutional lesson study. On one hand, clear roles and responsibilities can help structure the collaboration between the institutions, while on the other hand, they can lead to imbalance and hinder the teachers from developing shared didactic knowledge. It is crucial to acknowledge that the teacher conducting the research lesson at the lower secondary school should not be reduced to a mere secretarial role, as this would create an inappropriate imbalance. To foster a more equitable and engaged collaboration, it is important to recognize these dynamics and actively work to establish a collaboration where all lower secondary school teachers share responsibility for teaching the research lesson, while the teachers from both institutions are responsible for the research lesson plan.

6.5 Differences in teachers' expectations and students' performance

An important condition and simultaneously an obstacle to the establishment of a bi-institutional lesson study are the differences that occasionally appear between lower secondary school teachers' expectations of students' performance and what they actually observe during the open lesson. This difference is important for both lower secondary and upper secondary school teachers. They help identify which mathematical praxeologies students master and which didactic considerations need to be adjusted or implemented to help more students achieve the desired learning objectives during the open lesson. A particular aspect of this bi-institutional lesson study is that only one of the institutions, in this case, the lower secondary school, is able to have well-founded hypotheses and expectations about the students. This is because lower secondary school teachers have specific knowledge about these students that upper secondary teachers do not possess. This institutional asymmetry needs to be acknowledged, as lower secondary teachers are better suited to formulating these hypotheses and expectations. It is important to recognize that this asymmetry between the institutions is an inherent condition, given that the roles of teachers from the two institutions cannot be entirely symmetrical.

During the planning meetings, the lower secondary school teachers Alice and Max have specific expectations about how students will solve the task and the possible answers the students might provide. However, during the open lesson and the reflection meeting, we observed that the teachers' expectations did not align with students' actual performance in the research lesson. These differences can be beneficial for both the lower secondary and upper secondary school teachers, as they provide insight into which algebra tasks the students can solve and which tasks they find difficult.

At the planning meetings, Alice said: *"We expect that they should be able to set up expressions for the figure. In other words, the expectation is that if we have a figure with side lengths a and a, I expect them to be able to come up with a^2. We have also worked with Pythagoras, so I expect that as well."*

We see here that Alice expects the students to come up with a^2, arguing that they have previously encountered a^2 in a geometric context, namely the Pythagorean theorem. However, when students usually work with this theorem, they substitute the values of *a, b,* or *c* into the Pythagorean equation $a^2 + b^2 = c^2$, making it a type of task related to substituting in an algebraic model. In the selected task for the research lesson, the students are expected to rewrite an expression using the definition $a \cdot a = a^2$, thus making it a task related to rewriting or operating on an algebraic model. Alice expects that the students will connect the use of a^2 in one praxeology (about substituting in an algebraic model) to another praxeology (about rewriting or operating on an algebraic model), where the use of a^2 is related to rewriting an expression. This shows how didactic expectations can impact the selection of tasks and the didactic knowledge teachers can develop when these assumptions do not align with students' performance.

The lower secondary school teachers expect that the students can set up expressions for the area of the hexagon (see Fig. 2) and use the definition $a \cdot a = a^2$ to rewrite the expressions (see Fig. 3 for different expressions) and argue that they are equivalent. However, it turns out that these are the two aspects that the students struggle with the most. During the reflection meeting, the facilitator, Jack, who is not the author, asked Alice about her experience during the lesson. In response, she said:

Alice: We expected that they would be able to write down an expression with reasonable confidence.

Jack: Yes, could you (Alice) *say a bit about how it was for you to be up there today, of course, with a focus on what the intention of the lesson was?*

Alice: Yes, it was certainly not what I expected. It was not what I had expected compared to the first try. Well, the fact that they did not write down a^2 really came as a surprise to me… It actually surprised me a lot. I actually do not think there was a single person who had done something with squaring in the first individual round. I did not find anyone who had

These episodes illustrate the mismatch between the teacher's expectations about what the students master, and their actual performance observed in the research lesson. Based on her earlier expectations with the students, Alice expected that they would be able to set up an expression for the area of the hexagon. Since she knows the students best, compared with the upper secondary school teacher, it was more appropriate to consider her expectations during the planning meeting, but after these observations in the research lesson, it is also crucial to acknowledge that they do not align with students' real performance. This misalignment can provide valuable insights for teachers from both institutions to gain knowledge about the students' knowledge. However, such a discrepancy can also hinder the planned goals for the bi-institutional research from being met.

During the planning meetings, the teachers intended to structure the introduction in a way that would illustrate to the students why $a \cdot a = a^2$, by looking at the area of a rectangle with side lengths of *a* and *2a*, as they concluded that the chosen task would be too difficult without scaffolding, as also expressed by Henry: *"I think this question, specifically the area of the octagon, without scaffolding, is difficult. So, if you create that figure (the one with side lengths of a and 2a), it will be easier to move on to that one [pointing to the hexagon in Fig. 2] or something similar."*

The introduction to the research lesson began with a rectangle with side lengths of *a* and *2a*. The students were assigned the task of determining its area with a focus on the definition $a \cdot a = a^2$. This didactic decision, where the teachers begin with a simpler task, is believed to adequately prepare the students to successfully find an expression for the area of the hexagon and rewrite it. During the first trial of the lesson, it was observed that 4 out of 20 students achieved the learning objective, and two of these four later expressed that they were confused about why the expression for the area of the hexagon was what they had written. Similarly, in the open lesson, where the task was to find the area of the hexagon, almost none of the students provided a correct answer. An observation from the open lesson is that, with a greater focus on eliciting the definition $a \cdot a = a^2$, very late in the research lesson, approximately 5 out of 15 students were able to set up an expression for the area of the hexagon and rewrite it using the $a \cdot a = a^2$. However, there is no direct indication of whether these students have actually understood the rewriting or if they are applying the definition $a \cdot a = a^2$ because Alice had emphasized this heavily during the initial phase of the research lesson.

At the reflection meeting following the open lesson, the teachers discussed that this might be due to the way the introduction was conducted. In this context, Max said: "*I think one of the reasons why no one ends up including a^2 is that you (Alice) do not simplify the expression. You do not multiply the a's together. I think that could be it. It is simply because you do not start by multiplying the a's together, so there was not a single instance where there was anything, and that is why they do not go any further".*

During the reflection meeting, all the teachers, including Alice, agreed with Max's comments. They realized that a clear introduction of the problem (devolution in the sense of Brousseau, 1997) is crucial for a research lesson.

On a positive note, Alice also said, during the reflection, *"I was really surprised that four students came up and drew. I had hoped for that, but I did not quite expect it. It was great, I mean, that they came up"*. As usual in lesson study, moments at which differences are explicitly recognized between expected and observed student performance are crucial sources of new didactic knowledge, upon which also more daring research lessons can be built in later lesson studies.

7. Discussion and conclusion

We now summarize and discuss the main findings from the analysis of a first bi-institutional lesson study involving teachers from Danish lower secondary and upper secondary schools. The analysis identifies key conditions that facilitate both the establishment and implementation of this activity, as well as several obstacles that may hinder its establishment.

For the establishment of the bi-institutional lesson study, it is a condition that teachers from both institutions share some elements of a common model for the subject matter to be taught, as we saw that such a model makes the discussions and reflections among the teachers more concrete and productive. Engaging in shared didactic praxeologies can thus initially be facilitated by adopting or identifying shared views of the mathematics to be taught, that is, some elements of a common model. In our study, we saw how such a model allowed for relatively precise and productive discussions and collaboration.

For a bi-institutional lesson study to be established, teachers from both institutions must furthermore have a genuine, institutionally based interest in the didactic praxis at stake, which was the case most of the time in our experiment; however, this interest should also extend to previously developed, experience-based knowledge of that praxis. When this condition is lacking, it can lead to unproductive and defensive strategies on the part of the teachers normally in charge of the praxis, and (as we saw) particularly so for the teacher responsible for teaching the research lesson.

These conditions have the potential to be applied to other institutional transitions where transition problems might arise. Having a common model for the subject matter and fostering mutual interest in practical knowledge developed in another institution are likely to be important in any context where institutions need to collaborate on boundary objects, such as transitions between schools at consecutive levels, or the interaction of disciplines within a single school.

During the bi-institutional lesson study considered here, an institutional asymmetry between the two institutions exists and is an unchangeable reality. Lower secondary school teachers evidently should know more about their students' mathematical praxeologies. This, in principle, makes them more capable of forming valid hypotheses about the students' performance and knowledge during the planning phase of a bi-institutional lesson study. On the other hand, upper secondary school teachers teach at a higher level and are acutely aware of certain deficiencies in lower secondary school students' mathematical praxeologies, which they experience as these students arrive in their classrooms. This asymmetry can give rise to challenges in the collaboration unless very clear and shared hypotheses are formed during the planning phase of the lesson study, and shared reflections are developed on the extent to which these hypotheses are validated by the research lesson.

Despite these challenges, institutional asymmetry also has the potential for knowledge exchange, precisely as a consequence of the different backgrounds and experiences involved. It is evident that the upper secondary school teachers can learn about their first-year students' backgrounds by observing teaching and learning in lower secondary school. Conversely, with their stronger academic background in mathematics and experience of teaching at the next level, upper secondary school teachers can provide new perspectives on concrete mathematical praxeologies taught in lower secondary school, along with new didactic hypotheses for teaching them. In this way, the asymmetry can be leveraged to create a richer and more nuanced learning environment, where both institutions contribute something the other cannot. To realize this potential, it is important that both institutions are aware of the asymmetry and actively work to recognize and value the different roles and contributions each can offer. Viewing asymmetry as a resource is perhaps the essence of bi-institutional lesson study.

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Paper 3

Teachers' Development of Didactic and Algebraic Praxeologies through Bi-Institutional Lesson Study

Abstract

Transition problems in algebra between Danish lower and upper secondary school point to a need for teachers from different institutions to develop shared didactic and mathematical knowledge. The article presents *bi-institutional lesson study*, based on the anthropological theory of the didactic, and examines its potential for supporting teachers' development of shared knowledge of these transition problems. Despite structural asymmetries, the collaboration provided learning opportunities for both groups, helping teachers identify common challenges, such as students' use of guess-and-try methods when solving first-degree equations and difficulties with algebraic rules. Designing tasks requiring algebraic manipulation proved more difficult than expected. Teachers emphasized the importance of designing algebra tasks accessible to students with varying prior knowledge. The findings indicate a need for external inputs and further cycles of bi-institutional lesson study to achieve more substantial development of shared praxeologies, highlighting the limitations of what bi-institutional lesson study can achieve on its own.

Keywords: Algebra, lesson study, bi-institutional lesson study, institutional transition, the Anthropological Theory of the Didactic

Résumé

Les problèmes de transition en algèbre entre l'enseignement secondaire inférieur et supérieur danois mettent en évidence la nécessité pour des enseignants issus de différentes institutions de développer des connaissances didactiques et mathématiques partagées. L'article présente la lesson study bi-institutionnelle, fondée sur la théorie anthropologique du didactique, et examine son potentiel pour soutenir le développement, chez les enseignants, de connaissances partagées concernant ces problèmes de transition. Malgré des asymétries structurelles, la collaboration a offert des opportunités d'apprentissage aux deux groupes, aidant les enseignants à identifier des défis communs, tels que le recours des élèves à des méthodes d'essais-erreurs pour résoudre des équations du premier degré et les difficultés liées aux règles algébriques. La conception de tâches nécessitant des manipulations algébriques s'est révélée plus difficile que prévu. Les enseignants ont souligné l'importance de concevoir des tâches algébriques accessibles à des élèves ayant des connaissances préalables variées. Les résultats indiquent la nécessité d'apports externes et de cycles supplémentaires de lesson study bi-institutionnelle afin d'aboutir à un développement plus substantiel de praxéologies partagées, mettant en évidence les limites de ce que la lesson study bi-institutionnelle peut accomplir à elle seule.

1. Introduction

Could collaboration between teachers from two different and neighboring institutions be organized as an efficient means to work with concrete and well-understood transition problems for students as they move between the institutions? This study works with this question based on a specific form of teacher collaboration known as lesson study (Takahashi & McDougal, 2016), where teachers plan, conduct, observe, and reflect on a lesson together to enhance their shared knowledge about teaching and student learning. While lesson study has been widely used in different institutions and grades, it usually happens within a single institution. This leads to a curiosity about how this approach could be used to facilitate shared knowledge and practices across institutions, and particularly to facilitate the transition between them.

In the Danish context, research has recently identified and examined the relatively serious problems that algebra causes in the transition from lower to upper secondary school (Cosan, 2024). Gueudet (2016) emphasizes that algebra "has long been the "transition topic" par excellence, marking the frontier between elementary and secondary education" (p. 18). Cosan (2024) shows that the transition problem in algebra for the Danish students is related to rewriting algebraic expressions based on explicitly known algebraic rules such as the distributive property, and algebraic definitions like $a \cdot a = a^2$ – for instance, in the process of solving equations through algebraic rewriting. These algebraic challenges form the "concrete and well-understood transition problems" addressed in this study.

Gueudet (2016) also notes that many initiatives aiming to ease the transitions often focus on strengthening the relationships between primary and secondary teachers (p. 18). This motivates the present study to experiment with an adaptation of lesson study. We propose a new variant of lesson study, called *bi-institutional lesson study*, in which teachers from two neighboring institutions (here, Danish lower and upper secondary school) collaborate to plan, conduct, observe, and reflect on a lesson in the first institution. By engaging in a bi-institutional lesson study, the hypothesis is that teachers from both institutions can develop new and shared knowledge about how to address transition problems.

The following sections provide a more detailed presentation of a bi-institutional lesson study, based on what is already known about ordinary lesson study. We then introduce the Anthropological Theory of the Didactic, which is used in this study to model and analyse the mathematical and didactic practices involved in our experiment, and their institutional embedding; and, naturally, to formulate precise research questions about how they are affected by bi-institutional lesson study. Finally, the experiment and (elements of) its outcomes are presented and analysed in view of answering the research questions and generating new hypotheses.

2. Background on Lesson Study

Lesson study (also called *jugyou kenkyuu*), developed in Japan, is a practice-oriented and collaborative approach to teachers' professional development aimed at improving teaching and student learning. The most common form of lesson study occurs within a single school as a school-based teacher development program and typically lasts more than five weeks (Takahashi & McDougal, 2016). By connecting individual teachers' daily instruction with the school's vision for student learning, lesson study fosters both enhanced teaching quality and a stronger professional culture within the school or institution (Lewis et al., 2019).

According to Huang and Shimizu (2016), lesson study provides teachers with the opportunity to develop their professional competencies and gain a deeper understanding of student learning: "Several studies revealed teachers' development of their competence (including knowledge, skills, beliefs, self-efficacy, and disposition) … researchers documented teachers' changes in their mathematics knowledge and skills… understanding of student learning" (p. 399).

The lesson study process can strengthen professional collaboration among teachers, creating a framework for collective knowledge development. This collaborative structure serves as a systematic tool for knowledge sharing across schools or institutions. This structure enables teachers to share their experiences and reflections, thereby creating a collective culture of learning (Lewis et al., 2019). Moreover, Takahashi & McDougal (2016) point out that lesson study is often embedded in a "highly structured, school-wide project" (p. 514) that involves all teachers, ensuring that knowledge about teaching is developed and shared institutionally.

According to Takahashi & McDougal (2016), lesson study is organized into three phases that together form a lesson study cycle.

In the initial phase, *kyouzai kenkyuu*, a team of teachers studies the teaching materials, reviews curricula and textbooks, and identifies common student misunderstandings (Takahashi & McDougal, 2016). Takahashi & McDougal (2016) emphasize that this phase is crucial to the success of lesson study, as it requires "significant time" (p. 516) to provide teachers with deep insights into both subject matter content and students' thinking. They emphasize that omitting this "crucial phase of lesson study" (p. 514) can lead to superficial planning, thereby undermining the effectiveness of the subsequent phases in the lesson study process.

Seleznyov (2018) highlights that 63% of lesson study implementation outside Japan did not include *kyouzai kenkyuu.* As Fujii (2018) notes, this is an important part of the Japanese lesson study, and he emphasizes that teachers outside Japan who wish to implement lesson study must "both understand and successfully implement kyozaikenkyu" (p. 16). This makes *kyouzai kenkyuu* an essential prerequisite for maximizing the overall benefits of lesson study.

In the next phase, the team of teachers collaboratively designs a lesson addressing a specific, challenging mathematical topic, based on the thorough preparatory work conducted during the *kyozai kenkyuu.* This lesson, referred to as the "research lesson" or the "open lesson" (*kenkyuu jugyou*), is conducted and taught by one teacher from the team, while the other team members observe the lesson with a particular focus on students' thinking, learning, and behavior (Lewis, 2002). Additional educators may also participate in the observation of the research lesson (Takahashi & McDougal, 2016).

The process concludes with *kenkyuu kyougikai*, a so-called post-lesson meeting, where teachers engage in a structured discussion to share and analyse observations with a focus on how students responded to the lesson. The goal is to gain insights into the teaching and learning processes that occurred and to improve the lesson based on these discussions and reflections. Finally, the lesson may be taught again, and the process is documented in a report that includes the lesson plan (Takahashi & McDougal, 2016). As pointed out by Lewis et al. (2019), using a facilitator outside the team in the post-lesson discussions helps keep observers on track with their collected observations from the open lesson and the goals for the lesson.

A key figure in the lesson study process is the *knowledgeable other*, an external expert with expertise in both the academic content of the lesson as well as expertise in teaching and lesson study. This role is crucial, as the knowledgeable other contributes to the post-lesson discussion by highlighting key observations and linking the lesson to relevant research, thereby supporting the teachers' reflections and enhancing the growth of professional and didactic knowledge (Murata, 2011; Takahashi & McDougal, 2016). However, as emphasized by Seleznyov (2018), 55% of lesson study implementations outside Japan did not involve a knowledgeable other (p. 221), highlighting the lack of external expertise internationally.

Research has demonstrated that lesson study contributes significantly to improving teachers' content knowledge, particularly in specific mathematical topics such as fractions (as noted in Lewis et al., 2019), while also shifting their expectations for students' performance. Moreover, the collaborative nature of lesson study can change teachers' perspectives on the value of addressing mistakes and provide teachers with deeper insights into how students engage with the content of the teaching materials, thereby improving their professional practice (Lewis et al., 2019).

Huang & Shimizu (2016) emphasize that lesson study can lead to improvements in both classroom teaching and student learning outcomes: "These studies demonstrated three main types of evidence to document improvement of student learning … the third type of evidence is established on the basis of standardized tests that indicate gain of student's academic achievement" (p. 399).

Several studies document the positive impact of lesson study on student performance. For example, a study on lesson study focusing on fractions demonstrated a notable improvement in students' mathematical skills (Lewis et al., 2019). Similarly, a long-term case study of a school implementing a school-wide lesson study in mathematics found that the school's mathematics scores increased nearly three times more than the average in the district (Lewis et al., 2019).

3. Theoretical framework – The Anthropological Theory of the Didactic

Our work takes place within The Anthropological Theory of the Didactic (ATD), developed by Yves Chevallard (Chevallard & Bosch, 2020). This theory allows us to describe and analyze any human activity as a combination of practice and the corresponding knowledge and discourse about that practice (Barbé et al, 2005). ATD introduces the concept of praxeology to model human activities (Barbé et al., 2005).

A praxeology consists of two parts: praxis and logos. Praxis consists of a type of task (T) and a corresponding technique (τ). Praxis can be understood as various types of task, where "task" can be broadly interpreted to include mathematical tasks, such as solving an equation, and more general actions, such as brushing teeth or opening a door (Chevallard & Bosch, 2020). Logos consists of technology (θ) and a theory (Θ), where the former refers to discourse and knowledge that explain and justify the techniques, while theory is the knowledge and discourse that justifies and clarifies the technology (Miyakawa & Winsløw, 2019).

A mathematical praxeology thus includes mathematical types of task and corresponding techniques, as well as a logos about this practice, consisting of explanations of the techniques used, definitions of terms, rules, theorems, proofs, etc., that justify the technique applied (Miyakawa & Winsløw, 2019).

The focus here is on the mathematical activity – both what is done (praxis) and why it is done (logos). In contrast, didactic praxeology deals with the activities and the knowledge that teachers use to teach and convey mathematical praxeologies. Praxis in didactic praxeology might include planning lessons, presenting a task, or evaluating students' understanding, while logos contain theoretical knowledge about methods for teaching specific mathematical praxeologies and for evaluating students' praxeologies (Miyakawa and Winsløw, 2019).

The relation between these two praxeologies is central. Mathematics teachers' knowledge consists of mathematical and associated didactic praxeologies; the theoretical level of the latter includes, of course, also general ideas about teaching and learning.

As Miyakawa & Winsløw (2019) point out: "Mathematics teacher knowledge consists thus, foremost, of *didactic praxeologies*, but these are naturally inseparable from the mathematical praxeologies whose teaching they bear on, and can also be enriched (especially at the logos level) by related mathematical praxeologies" (p. 284). This emphasizes that although teachers' core knowledge is often didactic, this knowledge is closely connected to and depends on the mathematical praxeologies.

In ATD, teaching and learning mathematics are modeled as human activity conditioned and constrained by institutional settings. Teachers' praxeologies are shaped by and developed in the institutions they participate in. Thus, teacher knowledge needs to be developed beyond individual skills, in view of and by the institution as a whole (Bosch & Gascón, 2014).

4. Bi-institutional lesson study

This section introduces a new variant of lesson study: *bi-institutional lesson study*. It involves teachers from two connected and neighboring institutions, referred to as I_1 and I_2, collaborating to improve the teaching in I_1, particularly to ease students' transition to I_2. The collaboration is asymmetrical: I_1 teachers have in-depth knowledge of the students' current mathematical praxeologies, while I_2 teachers are familiar with the mathematical praxeologies expected when entering I_2. This difference provides an opportunity to develop shared knowledge among the teachers that can support students' transition.

Bi-institutional lesson study provides a framework for addressing these differences in expectations and for developing new and shared knowledge about students' learning and thinking. The collaboration allows teachers to explore each other's mathematical and didactic praxeologies, which are unique to each institution. Through this collaboration, teachers can gain an in-depth view of how these praxeologies differ and how such differences can impact students' learning and transition to the neighboring institutions.

I_2 teachers participate in all three phases of the lesson study: *kyouzai kenkyuu* (preparation and planning), *kenkyuu jugyou* (research lesson), and *kenkyuu kyougikai* (post-lesson discussion). They contribute with their experiences with the students from I_2, but it is important to note that there are no fixed guidelines regarding the role teachers from I_2 should take on. They do not initially function as facilitators or as knowledgeable others, but these roles may fall to them if they have the necessary expertise.

As in Japanese lesson study, a *knowledgeable other* also plays a central role in the post-lesson discussion in the bi-institutional lesson study. This person must have in-depth subject knowledge about the content of the lesson, as well as knowledge about the didactic and mathematical praxeologies of both institutions.

The facilitator role can be taken on by teachers from I_2 or an external person with knowledge of the didactic and mathematical praxeologies in both institutions. The facilitator's role is to promote a productive discussion among the participants.

What makes a bi-institutional lesson study unique is its cross-institutional nature, where collaboration aims not only to improve teaching in I_1, but also to build shared knowledge of students' knowledge, difficulties, learning, and transition problems among teachers from I_1 and I_2.

5. Research questions

Given the identified transition problems in algebra between Danish lower and upper secondary schools, and the potential of lesson study to develop shared professional knowledge, it is relevant to investigate how a bi-institutional lesson study can contribute to the development of teachers' didactic and mathematical praxeologies related to the transition problems. This paper, therefore, seeks to address the following research questions:

How can participation in a bi-institutional lesson study between teachers from Danish lower secondary and upper secondary schools affect their didactic and mathematical praxeologies? What are the specific potentials of this bi-institutional collaboration in relation to supporting students' learning of algebra?

6. Context and methodology

The implementation of the bi-institutional lesson study is primarily placed in a lower secondary school. Since there are known problems in students' transition from lower secondary to upper secondary school, particularly in algebra, it can be seen as an important task for lower secondary school to prepare students as well as possible. Therefore, the focus should be on the lower secondary level, where the initial difficulties with algebra arise and where the transition to upper secondary school needs to be prepared. This study focuses specifically on the collaboration between one lower secondary school (I_1) and one upper secondary school (I_2) in addressing these challenges. In Denmark, lower secondary school teachers are educated with a strong emphasis on pedagogy and didactics, whereas upper secondary teachers are educated with a substantially deeper specialization in mathematics and less emphasis on pedagogy and didactics.

The experiment (or invention) of this study was conducted in two Danish 9th-grade classes from the same school and involved six lower secondary school teachers (Leo, Julia, Anna, Mia, David, and Molly) and two upper secondary school teachers (Emma and Alex) from the same municipality. All names are pseudo-names. The teachers were divided into two teams: each consisting of three lower secondary school teachers and one upper secondary school teacher.

Each group planned and conducted three lesson studies on algebra in lower secondary school and observed the open lessons of the opposite group during the 2023/2024 school year. This meant that each teacher participated in a total of six lesson studies, which will be referred to as LS1, LS2, LS3, LS4, LS5, and LS6. All six lesson studies focused on algebra, specifically the setting up of algebraic models (algebraic expressions) in view of solving a given problem, as well as using the model to solve the problem through rewriting.

The lower secondary school involved in this collaboration performs significantly below average, both nationally and within its municipality, in terms of the number of students progressing to upper secondary school. This can create motivation among both teachers and school leaders to enhance (and promote) students' chances of successfully making the transition to upper secondary school.

The lower and upper secondary school teachers participated in all phases of the bi-institutional lesson studies, including:

1. Study of the curriculum
2. Formulation of goals for student learning
3. Lesson planning
4. First trials with a research lesson followed by a post-lesson discussion
5. Revision of the lesson plan for improvement
6. Open lesson followed by a post-lesson discussion

During the planning meetings, the author participated, while invited facilitators and knowledgeable others were present during the research lesson and post-lesson discussions. As part of the planning meetings, the author managed the meetings by keeping track of time and advancing with the purposes related to lesson study – for example, by ensuring that the lesson plan was completed on time.

In addition, during the planning phases, which encompass the study of the curriculum, the formulation of goals, and lesson planning, teachers not only developed the lesson plans but also collaboratively solved the tasks themselves. They discussed possible student solutions, anticipated common mistakes, and considered how these could be addressed during the lesson.

During the first trials, only the team conducting the lesson participated alongside the author. During the open lesson, one of the lower secondary school teachers taught the lesson while the others, including the teachers from the team and the other team, observed.

After each open lesson, a facilitator, an external guest from the university, leads the subsequent post-lesson discussion by ensuring that all teachers share their observations and select key points for deeper discussions. The knowledgeable other, also an external guest from the university, offered final comments by connecting teachers' observations from the open lesson and linking them to research. These comments related, among other things, to the lesson plan and the observed lesson, including how students worked with the tasks and shared their solutions.

6.1 Data collection

All planning meetings, research lessons, and post-lesson discussions were audio-recorded and transcribed in Danish. After the completion of the bi-institutional lesson studies, semi-structured individual interviews were conducted with all participating teachers. These interviews aimed to examine the teachers' own assessment of learning outcomes and the challenges they experienced in participating in the bi-institutional lesson study, both with colleagues from their own institution and with teachers from the other institution. During the interviews, the teachers were asked to recall concrete episodes from the entire bi-institutional lesson study. To support this, they were asked questions such as: *"Can you describe an episode where you felt you really learned something (that was beneficial)?"*, *"Can you mention a situation where the collaboration was particularly fruitful or challenging?"*, and *"How has it been to work with algebra in this project? Please mention specific episodes or situations where this became evident"*. Particular attention was given to episodes from the planning and reflection meetings, as these are the main situations where the teachers collaborated across institutions and reflected together on their teaching practice.

In addition to the audio-recorded planning meetings, research lessons, post-lesson discussions, and interviews, the lesson plans were also used as data material. These were included to identify the teachers' didactic intentions and goals for the open lessons, and to relate these to their reflections during the post-lesson discussions.

6.2 Data Analysis

To answer the research questions, the data analysis was carried out in three phases, guided by the Anthropological Theory of the Didactic, with particular focus on identifying and relating didactic and mathematical praxeologies.

In the first phase, all the data material was examined and sorted based on its relevance to the transition problem in algebra or to the bi-institutional lesson study. Specifically, data that did not address how the collaboration could contribute (or not) to the teachers' development of didactic or mathematical praxeologies related to this transition problem were filtered out. The selection criteria focused on whether the episode included either (a) references to specific algebraic content and its teaching across the two school institutions, or (b) indications or expressions of development of didactic or mathematical praxeologies among the teachers.

In the second phase, the interviews were examined to identify instances in which teachers explicitly described the bi-institutional collaboration as either difficult, challenging, or beneficial.

In the third phase, transcripts from the planning and reflection meetings were analyzed in relation to the interview data. The purpose was to identify episodes in which teachers' discussions either supported or challenged their reflections from the interviews. In this phase, the lesson plans developed for the research lessons were also included in the analysis, as they reflected the teachers' didactic intentions and goals for the lessons. By comparing the planned didactic intentions and goals for the research lesson with the reflections and discussions in the meetings, it was possible to explore how specific didactic choices in the lesson plans aligned (or not) with the teachers' expressed praxeologies, and how the outcomes of the lessons influenced further development or challenging of their mathematical and didactic praxeologies

To illustrate the data analysis process, one example was selected: during a planning meeting, the teachers collaboratively designed an algebraic task that could be solved by setting up a linear equation and solving it. They discussed how they expected students to approach the problem, as reflected in the lesson plan, which emphasized a guess-and-try method and the use of algebraic rewriting. However, in the subsequent reflection meeting, it became evident that the students did not solve the task as anticipated. This led to a discussion among the teachers about how to better support students in algebraic rewriting. In the interview, one teacher reflected on the differences between teachers' expectations and students' approaches, noting: "Perhaps we should look more into what makes sense for the students". This example shows how the analysis combined planning and reflection meetings with the interview data to illustrate the development of the teachers' didactic praxeologies.

7. Analysis and results

The following three sections examine how participation in a bi-institutional lesson study contributed to the development of Danish lower and upper secondary school teachers' didactic and mathematical praxeologies.

7.1 How differences in expectations led to changes in teachers' didactic praxeologies

Our data, presented below, highlighted how differences in expectations between lower and upper secondary school teachers became visible during the planning and reflection meetings in the bi-institutional lesson study, particularly in relation to how students solve first-degree equations. These differences contributed to the development of teachers' didactic praxeologies.

As mentioned, the teachers collaboratively designed six lesson studies (cf. Context). In LS5, they developed the following task *T* related to solving first-degree equations:

T: Lotte, Emil, and Daniella receive pocket money from their parents. Emil gets 5 DKK more than Lotte. Daniella receives three times as much as Emil. Daniella gets 125 DKK. How can we determine exactly how much Lotte receives in pocket money?

For the introduction of the task (the devolution in the sense of Brousseau, 1997, p. 31), the teachers planned to have a preparatory activity, where together with the students they formulate three algebraic expressions for Emil's (E), Lotte's (L) and Daniella's (D) pocket money:

Lotte's pocket money: L

Emil's pocket money: $L + 5$

Daniella's pocket money: $3 \cdot (L + 5)$

The students were then asked to investigate: "*How can we determine exactly how much Lotte receives in pocket money when Daniella gets 125 DKK?*".

The learning goal was to enable the students to "multiply into parentheses and solve an equation using inverse operation, and for the students to realize that the guess-and-try method has limitations".

The motivation and background for focusing on this topic were explicitly stated in the lesson plan:

> Many students feel that equations are a difficult topic, and this is something we [teachers] recognize from teaching, both in lower and upper secondary school. Many students have a limited understanding or perception of what an equation is. Many use the guess-and-try method and become discouraged when this approach is insufficient.

During the planning meeting, the teachers agreed on the equation $3 \cdot (L+5) = 125$ for students to work with. These discussions revealed differences in expectations about how students approach equation solving. While all teachers acknowledged that students often use a guess-and-try method, the lower secondary school teachers considered it as a sufficient technique, whereas the upper secondary school teachers emphasized the importance and difficulty for students to solve first-degree equations by rewriting them. To address this, in these discussions, the teachers shared a common didactic task: to design an algebraic task aimed at guiding students away from the guess-and-try method and toward using algebraic techniques like the distributive property.

The lesson plan for LS5 stated: "We [the teachers] deliberately constructed the task so that the solution to the equation was not an integer and not a number less than 10, which makes the guess-and-try method weaker".

During the planning, when the teachers discussed their expectations for how the students would solve the equation $3(L+5) = 125$, the lower secondary school teacher expressed that she did not believe the students would be able to guess the solution directly $\left(L = \frac{125}{3} - 5\right)$, since it is a non-integer less than 10, it makes it more difficult for students to guess the solution. Although the teachers want to design a task where the solution is not an integer and, in this way, make the guess-and-try method weaker, they still expect that the students will start with values like $L = 10, L = 20,$ and $L = 30$ and work their way toward the solution through a guess-and-try method. This reveals a contradiction: while the lesson aims to promote algebraic rewriting, the lower secondary school teacher still expects students to use guess-and-try, even though the solution is a non-integer. In contrast, the upper secondary school teacher assumes that the non-integer solution will make the guess-and-try method unlikely.

During both implementations of LS5, most of the students used a guess-and-try method to find an approximate solution, while only three students solved it orally and expressed the solution as a decimal number (e.g., "What must L be for $(L+5) = \frac{125}{3} \approx 41.66$?"). So only three students approached an algebraic solution to the task, although their reasoning was expressed orally. In the subsequent reflection meeting, the teachers, supported by the knowledgeable other, agreed that this low number indicated a lack of didactic support for most of the students. The issue is not the few who already possess some knowledge of algebraic techniques, but the many who do not. This led to a discussion on the need for didactic techniques to better scaffold the remaining students.

As Mia, highlighted in her interview:

> It has been rewarding to gain insights into my expectations and theirs [upper secondary school teachers], which were not exactly the same …Their expectations of what the students can do… at the upper secondary school, they expected the students to know more…they had higher expectations than I do.

In the first implementation of LS5, the task used 100 DKK as Daniella's pocket money. When students solved it easily by using a guess-and-try method, the teachers changed it to 125 DKK for the open lesson, aiming to reduce the guessing method. However, since both solutions are non-integer solutions, the changes made by the teachers were unnecessary. In the reflection meeting, the upper secondary school teacher noted that although students solved the revised task more slowly, they still did not use algebraic manipulation.

This outcome highlighted a didactic challenge: although the task was intended to promote algebraic rewriting, its structure allowed students to avoid it entirely. In the open lesson, students worked with equations of the form $A(x + b) = C$. The structure of this equation allows it to be solved by working backwards, which we observed students doing in the above-mentioned example $3(L + 5) = 125$. This issue arose despite the teachers having anticipated possible students' strategies and prepared the task introduction. However, because the task did not start explicitly with algebra and could be solved by working backwards $\left(\frac{125}{3} - 5\right)$, students saw little reason to engage in algebraic manipulation. The teachers had assumed it was an algebra task but they had not fully considered how students would actually approach the task, or whether algebra would appear meaningful or necessary from the students' perspective.

Although the teachers discussed several possible equations during their planning, they chose an equation of the form $A(x + b) = C$, as their experience suggested that equations like $A(x + b) = Cx + d$ or $A(x + b) = C(x + d)$ would be too difficult for students to solve. But if the goal is to bring the distributive property into focus, as the teachers intended, then equations of the form $A(x + b) = Cx + d$ or $A(x + b) = C(x + d)$ are likely necessary to address.

Through the open lesson, however, the teachers realized that such equations are necessary if they want to make the algebraic manipulation required for solving the equation. This realization marks a development in their didactic praxeologies.

They also recognized that designing tasks that merely allow algebra is not enough. If students are to engage with algebra meaningfully, the task must be structured so that algebraic manipulation is not just possible, but advantageous or even necessary. This might be achieved through equations of the form $A(x + b) = Cx + d$ or $A(x + b) = C(x + d)$. Otherwise, the natural emergence of a need for algebra will not occur. This reflection also revealed that both lower and upper secondary teachers had limited experience in designing algebra tasks that truly necessitate the use of specific algebraic techniques, such as the distributive property. Designing such tasks turned out to be more difficult than anticipated. Many students left the lesson confused and uncertain about the teacher's intentions and would likely continue to solve similar tasks without algebra, unless the task explicitly prevents the guess-and-try technique.

The pocket money task is a task that the teachers designed themselves. In comparison, the tasks in Figures 2 and 3 were more directly inspired by external sources. These externally sourced tasks also address rewriting algebraic expressions, but they function more clearly as algebra tasks that focus explicitly on algebraic techniques. This difference revealed that while the teachers collaborated effectively, the self-designed task lacked the algebraic clarity of the externally derived tasks, highlighting the challenge of designing such tasks from scratch.

The upper secondary school teacher noted that equation solving is not taught as a separate unit in upper secondary school but rather addressed when needed. This probably reflects an expectation that the subject is essentially dealt with in lower secondary school; as it transpired, this is not presently true, even if the curriculum and final exam could give that impression.

In his interview, Alex reflected on this collaboration:

> I think it has been interesting to discuss with the lower secondary school teachers what they believe the challenges are, and what I think … I have probably become clearer about what exactly students struggle with. Much of it, we probably knew beforehand, but we have discussed it and perhaps become more focused on what students struggle with in the transition from lower to upper secondary school … which is about mathematizing. That is when it becomes difficult for them [the students], as they move from concrete examples to working with letters. We experience again and again that many students just get left behind.

Alex's reflection highlights how the bi-institutional lesson study helped clarify the challenges students face when transitioning between the two institutions. While the collaboration led to shared assumptions, such as the idea that using non-integer solutions would reduce guess-and-try methods, these assumptions proved insufficient. Although the teachers had not yet succeeded in designing a task that necessitated algebraic manipulation, they became more aware of the complexity involved in doing so. In this way, their reflections marked a modest but meaningful step in the ongoing development of their didactic praxeologies.

7.2 Development of teachers' didactic praxeologies about lesson structuring

During the reflection meetings, teachers discussed how tasks can be presented to engage as many students as possible. A development in lower secondary school teachers' didactic praxeologies is their increasing emphasis on how tasks are presented (the devolution in the sense of Brousseau, 1997, p. 31) within a lesson. In particular, teachers reflect on how the structure of the constructed task and the lesson can support students' development of their mathematical praxeologies related to algebraic modeling.

For example, Leo says:

> Perhaps there is also something about starting [the lesson] with the very simple. I think we already do that, but I have definitely been reinforced in the idea that it is okay to make it concrete … starting with tasks that have a low entry point and then gradually building up. But perhaps being even more aware that there are students who are left behind, and I know they did not grasp it.

This reflection highlights how lower secondary school teachers recognize the importance of designing tasks that all students can at least undertake at some level or to some extent, regardless of their varying levels of prior knowledge ('low entry point'). This point has the potential to lead to a shift in the teachers' didactic practice related to task design, but at the risk of evacuating the need for students to advance into algebra. Specifically, the teachers recognize the importance of designing tasks that enable most or all students to develop their own methods, and the non-trivial character of this didactic task. This shift may also lead to changes in the teachers' view of their own role in teaching. At the same reflection meeting, Anna noted the importance and difficulty of "letting go so that they [the students] have to investigate and explore and come up with methods and ideas to reach an answer". These meetings, where such technical challenges are identified and discussed, also facilitate change in the teachers' didactic technology (θ'), where the focus moves from what teachers should explain to how they could facilitate student activity, while keeping some control of its direction. As Mia reflected in her interview: "it is a challenge to let go of the control and see where it leads … how can you keep it controlled while still allowing the students to actually contribute knowledge and participate in the lesson".

An analysis of the three lesson plans for LS1, LS2, and LS3, involving Emma, Leo, Julia, and Anna, indicates a development in the teachers' didactic praxeologies, particularly regarding how they structure the lessons to support the students' algebraic modeling.

In the main problem for LS1, the students had to determine how many edges existed in a structure of connected rectangular bridge segments (Figure 1).

Problem 1: How many edges are there on a bridge with 15 bridge spans?

Problem 2: How many edges are there on a bridge with an arbitrary number of bridge spans?

Figure 1: The main problem for LS1

While students successfully drew the bridge with 15 bridge spans and counted the edges (where an edge corresponds to a side of the rectangle) in problem 1, many struggled to generalize algebraically in problem 2. During the research lesson, many students said: "I can count, but I simply cannot figure out a formula for it". This illustrates a core challenge in students' mathematical praxeology related to algebra – moving from counting to an algebraic expression.

Emma emphasized in the post-lesson discussion for LS1 that their hypothesis regarding the development of this lesson was that: "if we do this enough times, we will generally strengthen the students' sense of algebra and the sense of abstracting from the concrete [numbers] to the abstract letters." However, the reflections from LS1 challenged this assumption. The students' difficulties with algebra tasks were not due to a lack of practice but were related to the specific way the lesson was organized. In particular, the transition from a concrete counting task to the formulation of an algebraic expression posed difficulties for many students. This highlighted the need to reconsider the lesson structure, including the didactic techniques used, especially how task design and teacher guidance could better support students in developing algebraic techniques.

In the main problem for LS2, the goal for the students was to gain greater familiarity with translating written instructions into algebraic expressions and rewriting these using the distributive property. In the problem named "Think of a number" (see Figure 2), students performed a sequence of arithmetic operations before being asked to express the process algebraically and then rewrite the algebraic expression.

1. Think of a number and follow the commands. Do this three times with three different numbers.

Command	Number
A: Think of a number	
B: Multiply the number from A by 2.	
C: Add 4 to the number from B.	
D: Multiply the number from C by 5.	
E: Divide the number from D by 10.	

2. Write an expression that corresponds to the commands in the table, where n is the number chosen in command A. Rewrite the expression to make it as short as possible.

Figure 2: The main problem for LS2

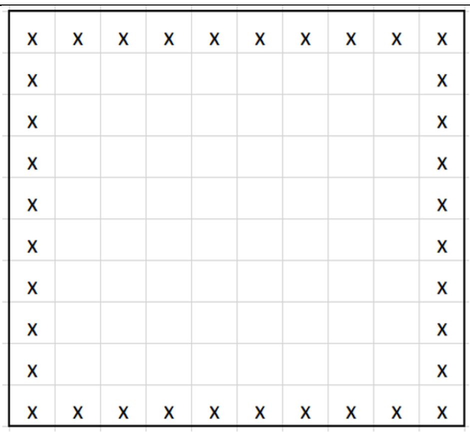
Leo refers to this in the reflection meeting, stating that "rewriting and simplification of the expressions posed significant challenges, which we anticipated. However, what I have gained is a clearer understanding of where we need to focus our efforts in order to move forward". This shows an initial awareness of the students' mathematical praxeology. However, the reflection remains relatively vague and does not yet lead to concrete didactic conclusions. This demonstrates how ongoing engagement in the bi-institutional lesson study might be necessary before teachers can develop more detailed analyses of students' mathematical praxeologies and how an outside perspective could be valuable in refining such insights.

Emma, on the other hand, noted that students seemed hesitant when it comes to algebraic rewriting and simplification: "In part three [in the research lesson], they [the students] were really unsure, 'what are we supposed to do?' … Maybe this group needed more instructions, but we had no way of knowing in advance".

This reflection shows how her didactic praxeology evolved towards more deliberate support of students' algebraic rewriting. Initially, Emma observed that students were uncertain about how to proceed with the task of rewriting the equation step-by-step, as instructed by the teacher.

Based on this observation, Emma suggested that, before having students engage with the task, it would have been beneficial to first show the step-by-step process of solving a first-degree equation. This reflects a shift in her didactic approach, from assuming that students would grasp the process on their own, to recognizing the need for more explicit guidance before they were expected to solve the task independently, pointing to an emerging awareness of students' algebraic praxeologies while leaving the form and timing of such guidance as an open didactic question.

In the main problem for LS3, the students had to identify a number pattern from a geometric situation and then set up an expression based on that pattern, which they had to solve as a first-degree equation. Concretely, the task required students to determine the number of marked squares along the perimeter of a square grid (see Figure 3). Initially, students worked with a specific numerical case (10 marked squares on each side of the square) and were then asked to generalize using algebraic expressions. Finally, they were given a total number of marked squares (352, see Figure 3) and asked to determine the side length of the square by solving a first-degree equation.



1. How many marked squares are there along the perimeter? How can you calculate that? How would you explain how you calculated it?
2. We now say that the side length of the square is *n* squares. Can you write an expression with *n* that can be used to calculate the number of marked squares along the perimeter?
3. We have a square with 352 marked squares along the perimeter. How many squares are there in the side length of the square? (Hint: use your expression to set up an equation and solve it)

Figure 3: The main problem for LS3

By using tasks with a low entry point that can be solved in various ways, teachers can include more students in the teaching. This focus on differentiated and open tasks is further nuanced through the bi-institutional collaboration, where upper secondary school teachers set expectations and contribute ideas on task devolution.

During the planning, the teachers discussed how to structure the task. For example, in LS3, the main problem began with question 1, which was designed to be solved at different levels. Some students counted the marked squares along the perimeter, while others formulated arithmetical expressions for the number of marked squares along the perimeter.

Anna noted in the reflection: "It was great to see that all [the students] were engaged at the first step [question 1]. They just got started. Some counted the marked squares, while others came up with different methods, like '8+8+10+10'".

Emma emphasized in the reflection meeting

> The students engage at the low entry point. They are quicker at setting up the expressions than they have been before. I also noticed that we had slightly changed the introduction regarding discussing 'what is n', and the way it was introduced today – as the number of squares in the side length – made the students grasp it extremely quickly compared to what they had previously learned.

Thus, both lower and upper secondary school teachers began to develop shared didactic knowledge about how to design lessons and tasks that enable most students to develop their algebraic knowledge.

7.3 Challenges for teachers in probing students' mathematical praxeologies

During the planning meetings for all lessons, the teachers identified a shared didactic type of task: how to support students in describing and justifying the techniques they used (individually) to solve an algebraic task. This didactic task was also emphasized in the lesson plans, such as in LS1, where one of the focal points for observation was whether students shared their thoughts, described and justified their techniques, and listened to each other's descriptions.

During these lessons, the teachers observed that students often struggled to both present their solutions orally and to justify their techniques. This highlighted the challenge of implementing didactic techniques for probing: how to ask questions that guide students toward describing and justifying their techniques.

The teachers' discussions during the reflection meetings show the development of a didactic technology about the need to moderate students' presentation of techniques involving probing students to make mathematical technology.

During the reflection meeting for LS1, the facilitator commented on that:

> One thing is that students need to say something, another is what they say. They need to speak mathematics using mathematical language. Today, we observed some students … say something, but it was not entirely mathematical. Is there anything that can be done to help them in this process, not just to come up [to the board] and say something, but rather, how can we support them in achieving a more mathematically precise language?

The facilitator's comment reflected an emerging theoretical view that probing students requires both attention to what students say and how they say it.

The difficulty in probing students became a theme in the reflection meetings. While the lower secondary school teacher found it challenging to facilitate the probing process during the open lessons, the other teachers struggled in the reflection meetings to provide concrete suggestions that could support students in explaining and justifying their techniques. While they acknowledged the importance of having students share their thoughts, whether in groups or at the board, determining how best to support the students' discussions remained a challenge. This difficulty was not limited to lower secondary school teachers; upper secondary school teachers also expressed similar concerns, emphasizing that the issue persisted across institutions. This shared experience indicates limits in the teachers' didactic techniques, prompting a need to expand their praxeologies. Consequently, external input, such as a knowledgeable other or a facilitator, may be necessary to support the development of their didactic praxeologies related to probing.

Despite these challenges, the reflection meetings also help teachers realize that fostering students' reasoning requires not only effective probing but also a focus on summarizing.

As Emma noted during the reflection meeting for LS2:

> Sharing thoughts is also something we have discussed a lot about in upper secondary school. Reasoning is essential, so how do we emphasize it in a product that they [the students] have to create? … If the focus is to be on reasoning, then it is about valuing that and letting something else take a back seat.

At the same time, teachers realized that summarizing had often been deprioritized in their teaching: As David remarked in an interview:

> Paying attention to wrapping up and following up - you can not spend too much time on it. It is always easy to tack on something afterward, but it is really difficult to actually do it, and you cannot do it in the next lesson. You have to do it right then and there.

In this way, the teachers recognized the importance of the didactic technique of summarizing to help students articulate their descriptions and justifications of the techniques they use to solve tasks more explicitly, although they still found it challenging to implement effectively.

8. Discussion

As shown in the analysis, both Danish lower and upper secondary school teachers participating in a bi-institutional lesson study became aware of how difficult it is to design tasks (for instance, word problems in algebra) that necessitate algebraic rewriting. The findings related to LS5 show that although the designed task for this lesson aimed at engaging students in algebraic rewriting, such activity did not occur among the students. If students can solve these word problems without algebraic rewriting, they tend to do so. These experiences were valuable for the teachers from both institutions in relation to how difficult it is to design tasks that require algebraic rewriting.

Designing algebra tasks involving word problems can be difficult for teachers for various reasons. Teachers often have limited experience in task design and are frequently inspired by textbook material, exam tasks, and similar resources. The findings of this study confirm this, as teachers from both institutions ended up designing what was intended to be an algebra task, which in the end had very little to do with algebra at all. This may also be attributed to the fact that teachers, especially those from I_1, often design tasks based on their knowledge of students' existing knowledge and misconceptions. Their expectations about how students will solve a task influence the design process, often resulting in tasks that can be solved without algebraic techniques.

Despite LS5 being a lesson that, in many ways, proved somewhat challenging and even, at times, unsuccessful, it still serves as a valuable example of how teachers can sometimes learn more about the students and teaching algebra when the lesson does not go as planned.

A central aspect of the bi-institutional lesson study is its asymmetrical nature. This asymmetry relates not only to the different roles and the individual knowledge of the participating teachers but also to the institutional praxeologies they are part of. Teachers from I_1 have experiential knowledge about their students' current algebraic praxeologies and difficulties, while teachers from I_2 have experiential knowledge of the algebraic requirements and students' degree of honoring them when entering I_2. These differences make the collaboration asymmetrical as the shared goal is to improve teaching in I_1 to prepare students for success in I_2.

In the Danish context, we could expect this asymmetry to be perceived as hierarchical, since the upper secondary school teachers come from a different and higher-level institution and enter the lower secondary school context. We could hypothesize that they might act as relative "experts" in mathematical praxeologies, which could empower them to instruct or even correct teachers from lower secondary school. Also, the fact that there are three lower secondary and one upper secondary school teacher collaborating could reinforce this asymmetry. Such an asymmetry could potentially weaken the collaboration if it leads to lower secondary school teachers feeling evaluated or if upper secondary school teachers assume a dominant role. The whole enterprise could also be seen by the lower secondary school teachers as focusing overly on the needs of the rough half of their students who eventually enter upper secondary school. What we see in the findings regarding the collaboration between teachers from the two institutions, however, is that this hierarchy did not materialize; both institutions contributed meaningfully to the collaboration. One might even say that the lower secondary school teachers were regarded as experts, as it was both their institution and their students, while the upper secondary school teachers were, so to speak, on unfamiliar ground, and thus had a greater potential for developing their didactic praxeologies. The findings also revealed that teachers from both institutions struggled with designing tasks in algebra, which provided both groups with an opportunity to develop new and shared didactic praxeologies.

An additional limitation of the bi-institutional lesson study can be the difference in the teachers' educational backgrounds. In Denmark, which is the case we are studying, lower secondary school teachers often have a broad pedagogical education with less specialization in mathematics, while upper secondary school teachers have a higher degree of subject-specific knowledge and specialization in mathematics. This difference can lead to challenges in collaboration, especially if the teachers fail to utilize their complementary backgrounds. For example, upper secondary school teachers' deep mathematical knowledge may come into conflict with the lower secondary school teacher's more pedagogical approach, which can create challenges in collaboration, such as disagreements about teaching areas or focus points regarding the teaching of algebra. None of these aspects was observed in the collaboration we studied. The empirical data and analysis from this study show that although the asymmetry exists, it did not result in a one-sided transfer of knowledge from upper secondary to lower secondary school teachers. The upper secondary school teachers developed their didactic praxeologies through collaboration, particularly by collaborating with the teachers from lower secondary, and the difficulties the students from lower secondary face. For instance, in LS1, the teachers became aware that students struggle to move from a concrete counting task to algebraic generalization, and that this transition requires deliberate task structuring and teachers' guidance. In LS3, both teachers saw how tasks with low entry points and with multiple strategies to solve them can engage all students. In LS5, they realized, through students' continued use of guess-and-try, how difficult it is to design an algebraic task that students cannot solve with a guess-and-try method, which they tend to do if the task can be solved with this technique.

The upper secondary school teachers involved in the bi-institutional collaboration expressed an interest in developing knowledge about the algebraic praxeologies in lower secondary schools, particularly in the 9th grade. This is evident from the interviews, where an upper secondary school teacher reported gaining knowledge about the difficulties that lower secondary school students face when solving first-degree equations through algebraic rewriting. Given these difficulties and the lack of algebraic knowledge among the students, teachers from both institutions shared a common goal: to support students' learning in algebra.

One limitation of this bi-institutional lesson study is that it took place in 9^{th} grade (final year of lower secondary school), while addressing the transition problems earlier, in 7th or 8th grade, could be more effective, as it would give teachers more time to address the challenges before students face the transition problem. However, this was a choice, as there are also reasons for placing the bi-institutional lesson study in 9th grade — for example, students are more mature, and upper secondary school and algebra are closer at that point than in earlier grades. Still, recognizing a problem is not enough if there is limited time to act on it, which an earlier bi-institutional lesson study might have provided.

We also observed in the analysis that teachers from upper secondary school suggested and used tasks from lower secondary schools' final exams as a starting point for their lesson studies, rather than introducing tasks from I_2. Had the latter happened, it might have reinforced the asymmetry in an unproductive manner. This leads to a broader hypothesis: For bi-institutional lesson study to be effective, it appears reasonable to suggest that I_2 teachers could engage with and utilize resources from I_1 before introducing their own. This approach can help limit the structurally embedded asymmetry and can serve as a motivator for collaboration among institutions if a shared transition problem is acknowledged.

It is also worth reflecting on the specific design of the bi-institutional lesson study. In this case, I_2 teachers observed lessons in I_1, given by its teachers. This choice gave them knowledge about teaching and students' mathematical praxeologies in I_1. However, bi-institutional lesson study could also be designed differently, for instance, by inviting I_1 teachers to observe an open lesson in I_2. One potential benefit of this alternative design is that it could reinforce both groups' view of the transition problem as a shared responsibility rather than solely their own. This could reduce any feeling of asymmetry in the current model, where primarily I_2 teachers observe I_1. Although our findings indicate that no clear dominance of upper secondary school teachers over lower secondary school teachers occurred in the collaboration, it is important to recognize that the underlying assumption of the bi-institutional study could itself be seen as a form of asymmetry.

Moreover, by observing lessons in I_2, I_1 teachers could witness the existence of algebraic transition problems, confirming that these challenges are not just perceived by the upper secondary school teachers. This might increase the motivation and relevance of the collaboration. On the other hand, a challenge might be the difference in mathematical background, which could limit I_1 teachers' contribution to the lesson planning and reflection. Nevertheless, observing the open lessons remains valuable, especially if scheduled early in the first year of upper secondary school, where the connection to lower secondary school and the transition problem are most apparent.

Our study has obvious limitations. First, it is small-scale and exploratory, involving a limited number of teachers in a specific context over one school year. It is carried out in a specific (Danish) context; at best, our observations there may function as evidentially supported hypotheses on the potentials of bi-institutional lesson study across neighboring institutions, which could be helpful for experiments in other, similar, contexts.

9. Conclusion

This study has examined how participation in a bi-institutional lesson study involving Danish teachers from lower and upper secondary school affected their didactic and mathematical praxeologies, and what specific potentials such collaboration may have in addressing the transition problem in algebra between the institutions.

Through the planning and reflection meetings in the bi-institutional lesson studies, teachers from both institutions developed shared didactic praxeologies regarding students' approaches to modelling non-algebraic problems with algebra and solving first-degree equations. Lower secondary school teachers observed that many students relied on a guess-and-try method despite prior teaching in algebraic manipulation. This observation was collaboratively discussed, leading to shared knowledge about students' algebraic techniques and how they solve algebraic tasks.

A challenge identified across both institutions concerned the design of algebra tasks, particularly those meant to involve the distributive property. The study highlighted that designing tasks that genuinely support students in reaching learning goals related to algebraic manipulation is a shared difficulty. When tasks do not explicitly begin with algebra, it becomes important for teachers to consider in advance whether the task actually requires algebra, in what ways, and whether students are likely to construct one or more algebraic models themselves. The bi-institutional lesson study provided an opportunity to address this challenge by exploring various ways to structure and design such algebraic tasks. As the examples show, success in the first trials is not guaranteed, and it is apparent the teachers from both institutions need more external input to advance from recognizing that it is a difficult problem.

Another outcome of the teachers' reflections was the recognition of the importance of designing algebra tasks that are accessible to all students, regardless of their prior knowledge. Lower secondary school teachers, in particular, emphasized the need for tasks that can be approached at different levels of student prerequisites, a concern that was reinforced through the bi-institutional collaboration.

Finally, the difficulty of encouraging students to describe and justify their techniques for solving algebraic tasks emerged as a theme in the reflection meetings. Although teachers from both institutions recognized the importance of encouraging students to share their mathematical techniques in groups or classroom discussions, they struggled to prepare and implement efficient didactic techniques for supporting this practice, and probably would need several more trials before reaching stable and efficient techniques for the didactic type of task. And again, more external input seems to be needed for them to collectively advance with this challenge.

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Paper 4

Transition problems and lesson study in secondary school

Abstract

Purpose

Transition problems between institutions are well-known and sometimes difficult to address in educational research. Lesson study is often described as a professional development tool for teachers within a school institution and so would not immediately be helpful to overcome transitions between two institutions. In this paper, we present the new format of "bi-institutional lesson study", developed and experimented with to explore the potential of this amended form of lesson study to develop teachers' knowledge of transition problems, and also methods and efforts to overcome them. We report findings from a year-long experiment with the format in Danish secondary school mathematics.

Design/methodology/approach

We conducted a year-long experiment with the bi-institutional lesson study (BILS) format, involving teachers from Danish lower and upper secondary schools. Within this format, teachers from both institutions collaboratively planned, conducted, observed, and reflected on the research lessons. Data was collected through audio recordings of planning and reflection meetings.

Findings

The experiment demonstrated that BILS enhanced teachers' awareness and knowledge of transition problems between the institutions. The Danish case, among other things, illustrates that teachers from upper secondary school mainly contribute knowledge about students' needs and task designs related to the algebraic skills required in upper secondary school. Lower secondary school teachers contribute by linking these needs and ideas to what 9th grade students have been and are being taught, mainly in arithmetic.

Originality/value

This paper introduces "bi-institutional lesson study" as a new format of lesson study, extending its use from a single school to collaboration across institutions. By engaging teachers from lower and upper secondary schools, the study presents a Danish case illustrating how BILS can develop teachers' awareness of transition problems and methods for addressing or working with them. This contribution offers both a practical model for teacher development and a novel approach to tackling the well-known transition problem between neighboring institutions.

1. Introduction

Throughout modern human life, we experience transitions between institutions as we pass through pre-school nurseries, various school institutions and workplaces. Passing from one school institution to the next typically involves a change of physical environment, teachers, and fellow students. But more subtle differences may occur between the institutions, and be related to the subject matter being taught (like different theoretical levels, technical levels, methods, topics, etc.). These differences are usually less visible to non-experts but are known to be the source of serious transition problems and learning obstacles – especially for students who experience that a new institution requires knowledge that they did not meet or learn in the institution they came from. Such transition issues are intensely studied in the mathematics education literature (see de Abreu el., 2002 for an early collection of studies).

Concerning more concretely the transition to secondary school, Gueudet et al. (2016, p. 18) point out that "many of the proposals to smooth the transition or to increase the engagement in mathematics of students entering secondary schools are mainly based on strengthening the relationships between primary and secondary teachers, sharing teaching activities carried out in both institutions, and, particularly, promoting more open activities during the first years of secondary school." But to our best knowledge, lesson study is not commonly deployed with the explicit aim of addressing transition issues, most likely due to the predominantly "school-internal" setting of lesson study as it has so far been presented in the literature and practiced in schools (Fernandez & Yoshida, 2004, p. 8ff).

The structure of this paper is as follows: in section 2, we first briefly present a concrete transition problem related to algebra in Danish secondary school, which has previously been studied in high detail by the first author (Author, 2024), and has parallels in many if not all Western countries. This problem originally motivated us to develop a new lesson study(-like) format: *bi-institutional lesson study* (BILS), which we present in section 3, as a general format applicable to institutional transitions. In section 4-6 we present a case study from our year-long experimentation of BILS in the Danish secondary school context, focusing on the transition problem from section 2. Finally, in section 7-9, outcomes and more general perspectives emerging from the case study are discussed.

2. Institutional transitions in mathematics and the algebra problem

Considering institutions in the abstract anthropological sense of "a legitimized social grouping" that is "credited with making routine decisions, solving routine problems, and doing a lot of regular thinking on behalf of individuals" (Douglas, 1986, p. 46f), it is clear that schools form a very special species. They are set up by society to bestow knowledge that it deems necessary for some special segment of its members, often defined by age; indeed, they are to confer knowledge which enables their subjects to decide, solve problems, and think according to certain common routine patterns. Thus, primary, lower secondary and upper secondary schools equip their subjects (aged about 6-12, 12-15 and 15-18) with ways to solve a variety of routine mathematical tasks, such as determining the area of a circle, solve linear equations and calculate derivatives of functions – and to think and talk about circles, equations and functions in similar ways. These institutions are not isolated islands: individuals pass from one to the other. However, their internal life depends on norms and enculturation, the latter understood as "induction, by the cultural group, of young people into their culture" (cf. de Abreu et al., p. 194). Here, mathematical cultures of institutions may differ slightly, even if all are somehow drawing on scholarly mathematics (cf. Chevallard, 1985). Circles are taught differently in primary and secondary school: for instance, as figures that result from certain practices of drawing, as points with a fixed distance to a given point, or as pairs of numbers satisfying some equation.

When transitioning from mathematics in one institution to mathematics in another one, some support may be afforded for newcomers to reconciliate previous techniques, norms and so on with the new ones, while still adopting the new. This is what Bishop calls *mathematical acculturation* (in de Abreu et al., 2002, p. 193), at times giving rise to conflicts with previous enculturation.

Using the generic notation I for a school institution, the transition from one such institution I_1 to the next I_2 therefore implies both adaptation, extension and modification of learners' mathematical practices, their reasoning, and so on. The school institutions have similar role structures – teachers, mathematics teachers, students of some given grade, and so on. However, the problems, methods and ways of thinking are systematically different: I_2 is expected to build on but also extend and modify the elements achieved in I_1. The hierarchical nature of mathematics could make transitions $I_1 \rightarrow I_2$ ripe with conflicts or discontinuities (cf. Gueudet et al., pp. 5ff) if the expectations (of students, and of teachers) are somehow not met – especially if society foresee that all or a large part of the students of I_1 move on to I_2.

Author (2024) investigated such discontinuities in relation to two such neighboring institutions (I_1: Danish lower secondary school; I_2: Danish upper secondary school), focusing more specifically on the subject *algebra.* Simplifying things a bit, and leaving out theoretical details, the main discontinuity was found to be in the technical and theoretical norms related to *rewriting algebraic expressions*, for instance during the solving of linear equations. These norms were studied through official documents (like textbook exercises and national exams) and through tests aiming at gauging subjects' capabilities close to the transition in question. Let us consider, as an illustrative example, the equations $2x + 4 = 10, 2x + 4 = 11, 2x + 4 = 11 - x$. While superficially similar, statistics from the national grade 9 exam suggest that

- most students at this point could solve the first,
- much less than half could solve the second, and
- only a few percent could do the third.

Indeed, the first can be solved by trial and error with small integers, which can be done mentally by substitution, or by "backwards reasoning" ($6 + 4 = 10$, so $2x = 6$, so $x = 3$). The latter method works with the second example, but the vast majority of equations appearing in textbooks and exams of I_1 have small non-negative integers as solution, and so students struggle with the last step in because of arithmetic limitations. None of the methods work for the last, where some form of rewriting is necessary – essentially due to the unknown appearing on both sides of the equation (cf. Filloy and Rojano, 1989).

More roughly yet, while algebraic notation is introduced in I_1, it is mostly used to model and reason about structures involving (positive) integers, such as number patterns. Rational numbers and methods to rewrite algebraic expressions (based on algebraic laws) are rarely encountered and not required at the national grade 9 exam, except for occasional outlier items that almost no student can solve. Teachers of I_2 appear to be aware of this, as evidenced by a widespread practice of using "review exercises" aimed at acculturating newcomers with the algebraic rewriting procedures that soon become inevitable, for instance in the study of linear functions, analytic geometry and so on. Evidence from the B-level national exam, after two years in I_2, suggest that many students have still not completed the transition in terms of capabilities for algebraic rewriting (Grønbæk, Jessen and Winsløw, 2019): only about 25% of students succeeded to reduce the expression $(a+b)^2 - b \cdot (2a+b)$ to a^2. More recently, this and other symptoms of overall weaknesses in the Danish school system has led to both official recommendations and initiatives to strengthen the teaching of algebra in all of secondary school (cf. Ekspertgruppe, 2022).

3. Bi-institutional lesson study

As pointed out by Miyakawa and Winsløw (2019), any education system will provide its teachers with some form of *paradidactic infrastructure*, i.e. institutional conditions for their work besides (*para*) teaching. This work involves both day-to-day tasks like planning lessons, grading student assignments and relating to parents and colleagues, and more long-term goals like professional development and career advancement. The conditions governing these activities include many different objects, from ministerial laws and decrees to resources and physical space available for lesson planning. By contrast, the *didactic infrastructure* refers to the conditions for teaching, including classrooms, teaching resources, time frames and so on.

Of course, these conditions will be specific to some school institution *I*. Lesson study in Japan is indeed a school-based element of the wider paradidactic infrastructure for in-school teacher training (*konaikenshu*, cf. Fernandez & Yoshida, 2004). Lesson study is known to engage virtually all Japanese teachers in primary and lower secondary school (Lewis, 2016). As an activity, it is not *strictly* internal to a given school institution: not only does it frequently involve a *koshi* or "knowledgeable other" who may or may not be a teacher in the same institution (Takahashi, 2014), but it is not uncommon to allow or even invite external observers from other schools, local or national officials working with education, publishing companies and so on (Isoda et al., 2007, p. 20). Lesson study in Japan thus has a function which goes beyond the single school institution and in particular, it provides a two-way interaction between reforms and classroom practice.

Over the last two decades, LS has spread to other countries, including the United States and Denmark (Ding et al., 2024; Østergaard et al., 2021). As pointed out by Seleznyov (2018), there is no internationally accepted definition of lesson study. Some of the components that characterize Japanese lesson study are frequently missing in other countries. In particular, foreign adaptations have been implemented as an add-on to existing paradidactic infrastructure within some specific school institution, thus focusing on the *konaikenshu* aspect of the format. The activity is centered around a study lesson but has no wider implications or goals related to curriculum development, teachers' careers, or other institutions.

In this paper, the institutional interaction is put to the fore, given the transition problems outlined in the previous section. In what we will call *bi-institutional lesson study* (BILS), teachers from two neighboring institutions I_1 and I_2 engage in lesson study where study lessons are carried out in I_1 while deliberately focusing on known and concrete problems relating to the transition $I_1 \to I_2$. The goal is to provide teachers with opportunities to develop shared experiences of teaching in I_1, with a main goal being to identify opportunities and obstacles to prepare all students for the transition. In implementing BILS it is considered crucial to explicitly take into account the paradidactic infrastructure of both institutions and how they may differ, such as having different curricula, teacher education systems, organization of teachers' work, etc.

Teachers from I_1 and I_2 will have different roles in BILS. Teachers from I_1 occupy the roles teachers usually have in a lesson study. Teachers from I_2 are more akin to external partners, who in this case are particularly informed and interested about the transition and more concretely, what students need but lack at the entrance of I_2. The latter will need to familiarize themselves with resources and curricula from I_1 to identify which of these needs are or could be catered for in I_1, according to that institution's goals and means. Planning lessons together begins with negotiating concrete lesson aims on this background.

As always when lesson study is added to a given paradidactic infrastructure, an external partner is needed. In the case of BILS, it is evident that this partner needs to be thoroughly familiar with the infrastructure of *both* institutions, to contribute to formulate goals that BILS could reasonably undertake, based on a thorough (external) analysis of the transition problem focused on. For our concrete experiment with BILS, such an analysis was outlined in the preceding section. We now proceed to present the research question addressed in the present paper, and its methodology.

4. Research question and methodology

Besides introducing the theoretical construct of BILS, the main purpose of this paper is to present a selection of findings from observing one specific BILS, as part of a year-long experiment with the format. The findings pertain to the following research question: *what are the potentials and obstacles for BILS as a paradidactic infrastructure for teachers from neighboring institutions, when it comes to addressing a specific transition problem between the institutions?*

The findings presented in this paper come from a single case study, taken from a series of explorative case studies of BILS (cf. Yin, 2014). These BILS were carried out throughout one school year in the context of Danish lower and upper secondary school (neighboring institutions). After a one-day workshop organized by the authors to introduce the BILS format, two teams with representatives from both schools were formed. During the year, each team carried out three lesson study cycles, with the study lesson being taught twice and revised in between. The first author attended all sessions and collected the data (cf. below). In each of these six cycles, the last study lesson was observed by the other team as well as by experts of mathematics education from tertiary institutions.

The aim of the experiment was to test the hypothesis of BILS as a means to address the algebra problem outlined at the end of Section 2 (the specific transition problem). The outcome of each of the six case studies (corresponding to the cycles described) is of course not merely "positive or negative", but rather a collection of observed potentials and obstacles for this hypothesis. We chose to present one case because it reveals a significant and generalizable set of potentials and obstacles that were also observed in several other cases. Focusing on a single case also allows us to share essential details within the format of a journal article.

The descriptive part of the case study is based on transcribed audio recording of planning meetings, one study lesson and the corresponding reflection meeting, as well as on documents (lesson plan, handouts from study lesson) and pictures of students' and teachers' writing during the lesson. We identify potentials and obstacles (cf. research question) by confronting the description with the institutional analysis outlined in Section 2. We analyse contributions to the reflection and planning sessions, to identify generalizable potential and obstacles to planning and learning from a lesson, while drawing on two institutional perspectives, given that the lesson focuses on boundary objects from algebra (cf. Star & Griesemehr, 1989, 408). As always in a case study, the outcome is itself hypothetical (albeit in this case, confirmed by other cases from the year-long experiment, to be published in the first author's doctoral thesis).

5. Case study: context

The lesson was held in a grade 9 class at a municipal lower secondary school (I_1) of a smaller Danish town with medium to low level average grades at the national exam. Teachers at the school were in part motivated to participate by the relatively low rate of students advancing to upper secondary school (less than 50 compared to a national average of 70); whether as cause or effect, this state of affair was explained by school staff as related to a tendency for resourceful parents to prefer other schools in the municipality. Teachers from a nearby high school (I_2) represented the neighboring institution (cf. Section 2).

The team designing the lesson comprised three I_1-teachers and one I_2-teacher. This was their second study lesson. The main idea for this lesson was proposed by the I_2-teacher: to use a classical "think of a number"-trick, and have students explain the outcome based on algebra. The "trick" is as follows: "Think of a number, multiply it by 2, add 4, multiply by 5, divide by 10". The lesson plan organizes the lesson in terms of three tasks for the students, devolved by the teacher in the given order and with more informal wording:

1) try out the trick with three numbers of your own choice
2) set up an "expression with letters" to represent the trick
3) reduce the expression to explain the outcome of the trick.

During the study lesson, 17 students work individually with the first task and in pairs with the last two. The work with each task is supported by a handout (see Fig. 1) which the students fill and share (orally in part 1, by posting their handout on the board, wall or windows in part 2 and 3 (cf. Fig. 1).

Command	Number
Think of a number	
Multiply by 2	
Add 4	
Multiply by 5	
Divide by 10	
What do you get?	

Command	Expression
Think of a number	n
Multiply by 2	
Add 4	
Multiply by 5	
Divide by 10	
What do you get?	

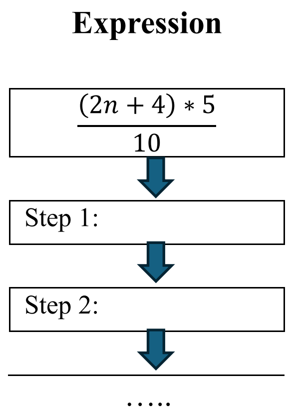

Figure 1. Tables given in student handouts for task 1), 2) and 3)

In the lesson plan, the teacher team anticipates that 1) will be easy and that at least some students will discover that the result is the "2 over the chosen number". In 2), they expect that some students will produce expressions with misplaced parentheses, and the plan exhibits some imaginative examples. In 3) they anticipate that some students will be unable to use the distributive law correctly.

6. Case study: observations and reflections across institutions

At the reflection meeting, the teachers first share observations related to students' varying success with the proposed tasks. To summarize: all students succeeded in filling the table for 1) shown in Fig. 1, most students struggled with 2) and 3) and in fact, only a handful of students managed to get a correct expression in 2), and to get $n + 2$ a the end of the table for 3) (shown in Fig. 3). In other words, only a few of the students got to experience the point of the lesson fully: namely, that algebra allows one to explain why the number trick always gives the initial number + 2. The rest struggled and failed at least partially to fill in the handouts. They could of course still have benefited from the plenary validation sessions.

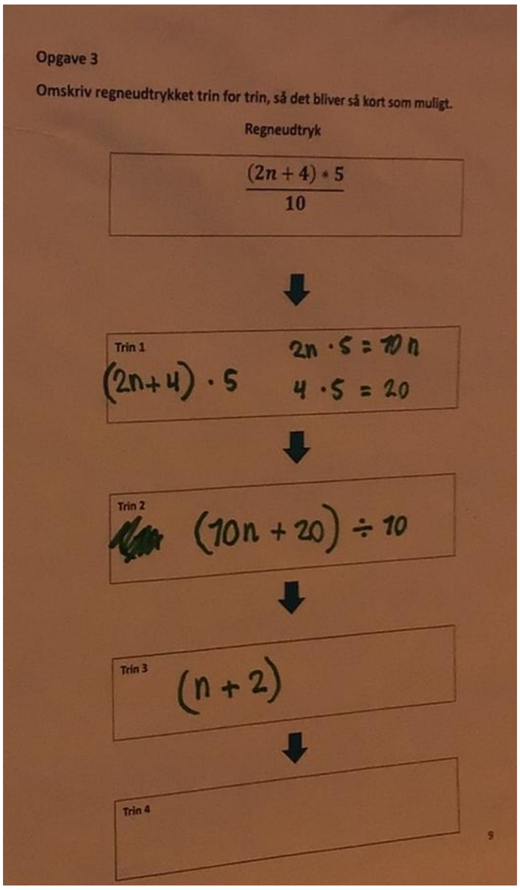

Figure 2: student writing on the table for task 3). The text on top says (in Danish): "Rewrite the expression step by step, so that it becomes as short as possible."

The study lesson was observed by the designing team, another team of similar composition, and three external observers (the first author and two teacher educators). We focus here on the reflection meeting and in particular the contributions of the 6 I_1-teachers and 2 I_2-teachers, as an expression of the teachers' interaction and take-aways from the lesson which illustrates several more general traits we observed in several other experimentations of BILS.

First of all, the teachers from both groups make a large variety of concrete observations, but their perspectives differ. For the lower secondary school teachers, the algebraic model is an abstraction of what they see as concrete, namely the arithmetic procedure embodied by the number trick:

> What I experience and see is that most or a big part of the students participate in part 1 and find three numbers they try it out with. They discover or get that the number they end up with is +2 (…) Then comes the barrier when vi introduce the variable *n*. (…) Many get stuck when we go from the concrete to the more general. (I_1-teacher who taught the lesson)

A colleague from I_1 agrees that many students could get through 1) and get a result, but points out that not many noticed that the result is "+2"; some "challenged themselves" with "high numbers" and then made calculation errors. This point is returned to here and there in the discussion, where it also appears that it was not the intention of the team responsible that this observation or hypothesis should be established for all before going on to 2) and 3). An I_2-teacher says:

> I remember we thought it could be a bit secret [that the answer is $n + 2$]. We observed at the first trial [of the study lesson] that some students quickly came up with $n + 2$ which was something to only take up later. (I_2-teacher who was part of the planning team)

One of the observing I_1-teachers who was not part of the planning team argues that the algebraic work in 2) and 3) could be motivated by a preceding observation of an apparent pattern ("+2"):

> So it's all about letting the students raise the question: "why is there a pattern". What can we do to let this curiosity come up for the students, so they get curious about "why is it always $n + 2$" and "why is there a pattern".

This is seconded by an I_2-teacher (also not part of the planning team):

> Maybe more would get more [students] onboard by doing a table with *n* and then write the numbers underneath. Then it could become very clear for them that *n* becomes 2 larger. Did you consider that? [*sic*].

It is not clear from the lesson plan what was foreseen in this regard. It merely states that "some students may already here [in 1)] discover the pattern, that is, the result is $n + 2$ (but in words, without symbols)."

The comments of high school teachers otherwise focus on the algebraic work related to 2) and 3), such as the following observation:

> Really many in the class wrote n^2 instead of $2n$ and said: "I multiplied by 2", but then they think of it as n^2 when it is said. And then I think, we had this also in some of the previous lessons where we really worked with a times a is a^2 so they got that this with a^2 is really important (…). Maybe they wrote n^2 but there were also really many who used a fraction line instead of division symbol. That was cool cause we had not expected that. So in some way they are onto something, I think… (I_2-teacher who was part of the planning team)

This refers to the (planned) common form of the algebraic model which the team proposes as a starting point for 3). The I_2-teachers preference for fractional notation is not further explained. We note here that rational expressions like $\frac{x^2-1}{x-1}$ are frequent and crucial in I_2, particularly in relation to functions and derivatives. In the present case the expression $(2n + 4) \cdot 5/10$ is just as useful, in view of the goals stated in the lesson plan: "[for students] To be able to set up algebraic models, using parentheses, [and] to use the distributive law and to divide".

While much of the I_2-teachers' attention is thus on details in the students' use of algebra, the I_1-teachers continue to focus on the distance between that and what the students are more familiar with:

> … The students are very result oriented. It's like when the answer is not a number, it frustrates the students, as they don't think they are done. When they get an expression then it's hard for them to know that they are actually done. It's hard for them to stay so long with

one task (…) so it's something with the culture here. (I_1-teacher who was part of the planning team)

Another I_1-teacher adds that to prepare for the centralized, final exam of I_1, the students do not have "time to nerd with one task and be curious (…) that provokes me all the time". Indeed, as was pointed out in Section 2, algebraic modeling and rewriting plays a very small part in this exam, and algebra is mostly about substituting (small positive) integers in expressions and equations. The I_1-teacher who taught the lesson pointed out an observation which could help to integrate this point of view:

> Another comment from [student name], which is good and general, was: "when we have reduced, it should give the same. He had not done the task [3)] but he had clear knowledge that even if he rewrote the expression, it would give the same when you put the number in. And that's a good knowledge to have." (I_1-teacher who taught the lesson)

In brief, the I_1-teachers focus more on connecting what they see as the "general" algebraic tasks back to the arithmetic calculations that the students are more familiar with. In task 2) and especially 3), this connection appears to have vanished for many students:

> I am a bit curious about this "what is it really that one should reduce/shorten/simplify". The boys struggled and were like "what is it after all that I am asked to do" (I_1-teacher who was not part of the planning team).

This sparks a discussion about the many difficult terms and explanations in the teacher's dialogue with students. For instance, an I_1-teacher considers that for some students, "reduce" is related to fractions, and that second-language learners in the class did not seize the special meaning of "expression" in the algebraic context. Another I_1-teacher notes that the posters (in Fig. 2) were impossible to read at distance, which made it even harder to follow students' oral explanations. That tasks 2) and 3) were overall destabilizing the students also became evident to the I_2-teacher from the planning team: "I did not think it was so complex, but I can see now there were many things for them to do". The teacher suggests they could have asked students to explain each step (for instance in Fig. 2) in writing.

To sum up, the teachers from I_1 and I_2 clearly offered complementary perspectives on a lesson which, while planned by representatives from both institutions, offered both expected and unexpected difficulties to carry out. It even reveals some uncertainty in the planning about how to link 1) with 2) and 3) – although the tasks (together) offer potential for doing so. The I_2-teachers realize that algebraic modeling and rewriting offers (as one says) "many things for them [students]to do", and they reflect on what could help students succeed with tasks 2) and 3). The I_1-teachers focus on connections with what students are otherwise being taught in I_1. For instance, one I_1-teacher says that having seen students struggle with 2) made him decide to work more intensively with the "arithmetical hierarchy" in his grade 5 class (i.e. work more with the order in which operations are to be carried out in a written expression like $2 + 3 \cdot 4$). A colleague made a similar connection to higher grades:

> What I got [from this lesson] is a clear indication of "where we are with this" (…) so in that way I have got an insight into where we should start working more in relation to this

mathematics and these law-like connections. Some [of the students] use, unknowingly, the distributive law. It's also something to work on. (I_1-teacher who taught the class).

This quote is also a rare example of (almost) direct reference to the goals stated in the lesson plan (and cited above). The final comment, given by a researcher and teacher educator, underlines the importance of attending to these goals, and to make the knowledge required for each step explicit to the students – for instance, by providing them with "a small text" to refer to, on the use of parentheses and the distributive law. Students should not only "inquire" (that is, explore questions) but also use and learn concrete and explicit mathematics which could be used later.

7. Implications and discussion

The concrete case of a BILS outlined above highlights certain potentials and obstacles for this format to address students' transition problems across neighboring institutions.

A primary potential is that teachers of I_2 achieve concrete knowledge of the practices of students and teachers in I_1. As not all students from I_1 proceed to I_2, they also get first-hand experience of the difficulties that attempts to reinforce a focus on a transition problem (here algebra) could meet in I_1, as the case illustrates very well.

An almost mirroring potential emerges for teachers of I_1, in the course of BILS-supported experiments with such attempts: during planning and reflection, they meet with input and observations from I_2-participants, which reflect the needs and shortcomings they experience in I_2, in some cases not only at the entrance but also later in the curriculum there. They are also challenged by the concrete observed challenges, for instance in relation to the students staying the entire lesson with essentially one problem.

An obstacle appears in both cases: the perspectives may be too different to enable a continuously developing dialogue. While this obstacle could conceivably decrease over time, it is also clear from the case that not all potentials of BILS depend on exchange; observations that lead to consequences for teaching in earlier grades (by I_1-teachers) or for teaching in I_2, do not necessarily have to be shared, but are still visible potentials.

At the reflection meeting in the case just examined, we repeatedly saw the difference in perspectives the two teacher groups bring to the meeting (in particular, different experiences, knowledge, and aims related to basic algebra). It is a basic potential of BILS to deploy and combine such perspectives on core teaching tasks related to the transition problems, which may even to some extent be caused by the total absence of shared paradidactic infrastructure (cf. next section).

It appears in general to be an important condition for BILS (involving two neighboring institutions) that some overarching concern or problem in an educational system can be traced to inconsistent or disconnected practices in the actual teaching of the two institutions. From the reflection session, it appears that teachers from I_2 mainly contribute knowledge about needs and task designs that are related to algebraic skills they require (and to some extent teach) in I_2. The I_1-teachers contribute by linking those needs and ideas to what 9th grade students are and have been taught, mainly in arithmetic. Observing commonly designed lessons leads to shared – albeit very partial and limited – new hypotheses on how to address the boundary objects at hand, whether in the teaching of I_1 or of I_2.

The links between observations are not necessarily shared or appreciated equally by all participants of a given institution – even with a synthesizing final comment as the one outlined in the case. And in fact, individual benefits from lesson study are also of potential local value. But impact on the wider transition problem would require dissemination of principal outcomes of such experiments through the separate paradidactic infrastructures of I_1 and I_2, and to the policy level (cf. for instance Ekspertgruppe, 2022). The involvement of two sets of educators appears, at least in our context, to be a necessary element to ensure such wider impact. This is further explained in the next section.

8. Further limitations on BILS

In many contexts, neighboring institutions (I_1 and I_2) are managed and operated by separate entities, for instance, within Ministries of Education or more local authorities. This leads to more or less separate paradidactic infrastructures of I_1 and I_2. This could be a real, practical obstacle to realizing BILS as regular professional development – outside of an experiment like the one implemented by us. It could mean that the initiative and funding must come from outside of the two institutions and their "normal" infrastructures. Our year-long experiment was funded from a grant obtained jointly by a university and a university college (as institutions responsible for teacher education related to I_2 and I_1, respectively). While some of the I_2-teachers had experienced lesson study in other university-led projects, the participation of both teacher education institutions was not only a resource but also a necessity in this case, both to obtain the funding and to introduce and legitimize the concept of BILS. This also touches on the question of the sustainability of this form of lesson study (cf. Østergaard and Winsløw, 2023); as soon as the external support expired, the BILS activity ended at the involved schools. BILS appears thus at present as a one-year experiment.

Another possible obstacle could reside in partially different aims and incentive structures of the two institutions. In our case, I_1 is a comprehensive school, while I_2 is only an option for some students. Teachers of I_1 – especially in grade 9 – are naturally focused on their students' success at the final exam (as also expressed in one teacher comment in the case above). In relation to the focus on algebra, we saw in Section 2 that students' success does not depend much (if at all) on their capability to set up or to rewrite algebraic expressions as involved in task 2) and 3) of the lesson. Such capabilities, on the other hand, are crucial in I_2 but continue to be missing in many (if not most) of the students there. Based on the different curricula alone, it would thus seem that engaging in BILS on this topic is primarily of interest to I_2 and its teachers.

However, I_1 as an institution in Danish society, depends not only on its students' success at the final exam, but also on the extent to which its students are prepared for life and education after graduation. Comprehensive schools (to which I_1 belongs, covering all 9 mandatory school years) are monitored in terms of grade averages and contributions to social mobility, where success to advance to I_2 is known by parents and economists alike to be a major factor. As mentioned in Section 2, recent efforts initiated by the Ministry of Education focus specifically on arithmetic and algebra, and teachers in both I_1 and I_2 are in fact strongly motivated to engage in it. This was apparent from the start of the project, as it is also visible in the event outlined in Section 6.

9. Conclusion

Neighboring institutions may function in more or less pronounced separation, and both transitions problems, as well as obstacles and potentials of BILS to address them, most likely increase with the degree of separation. In our case the teachers work in materially separate places, and have few if any occasions to ever meet – even if they teach the same students (before and after a summer break). The teachers from I_1 and I_2 have institutionally separate pre- and in-service training, with very little common content; they belong to different unions and could not get to work at the other institution without several years of supplementary training. Not only for students in the transition but also among the two teacher groups, one could really talk of different mathematical cultures (cf. de Abreu et al., 2002, p. 199); this is presumably often the case for neighboring institutions. Nevertheless, instances of how BILS enabled a fruitful exchange of perspectives did appear in our case study.

It appears sometimes as an implicit expectation among novice Western proponents of lesson study that the format would magically enable the teaching profession to develop "all by it itself", independently of other elements of paradidactic infrastructure. Our case and the rest of our experiment suggest that BILS cannot be assumed to realise such an expectation. Even in Japan, where BILS does not seem to exist, the paradidactic infrastructures set up by the Ministry of Education, regional and local authorities strongly favor lesson study and similar formats (Miyakawa et al., 2019); and external partners, including universities and teacher educators, are integrated in these infrastructures in many ways. The potentials revealed by our case study rest on similar cross-institutional efforts, and should be understood as largely relying on them.

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