EXEMPLIFICATION IN MATHEMATICS EDUCATION

Liz, Bills, Tommy Dreyfus, John Mason,
Pessia Tsamir, Anne Watson, Orit Zaslavsky

(Universities of East Anglia, Tel-Aviv, Open, Tel-Aviv, Oxford, Technion-Israel)

1. BACKGROUND

There is evidence from earliest historical records that examples play a central role in both the development of mathematics as a discipline and in the teaching of mathematics. It is not surprising therefore that examples have found a place in many theories of learning mathematics. Many would argue that the use of examples is an integral part of the discipline of mathematics and not just an aid for teaching and learning. The forum takes as its background both the variety of ways in which examples are construed within different theories of learning and the contribution that attention to examples can make to the learning and teaching processes. Consequently the forum can be seen as addressing issues at the very heart of mathematics education, both drawing upon and informing many other research topics. We argue that paying attention to examples offers both a practically useful and an important theoretical perspective on the design of teaching activities, on the appreciation of learners' experiences and on the professional development of mathematics teachers.

The importance of these ideas does not actually depend on the framework used for analysing teachers' intentions, nor on any terms used to describe forms of teaching, such as: 'analytic-inductive' or 'synthetic-deductive', 'traditional' or 'reform', 'rote-learning' or 'teaching for understanding', 'authentic' or 'investigative'. Issues in exemplification are relevant to all kinds of engagement with mathematics.

This paper positions exemplification on the research agenda for the community by giving a historical overview of the way examples have been seen in mathematics education; an account of associated literature; an exploration of how exemplification 'fits' with various perspectives on learning mathematics; accounts of issues relating to teachers' and learners' use of examples; and directions for future research.

2. WHAT IS A MATHEMATICAL EXAMPLE?

The word *example* is used in mathematics education in a wide variety of ways. This section offers a brief overview of the scope of our use of the term and points to some useful distinctions that can be made between different uses.

Examples in the form of worked solutions to problems are key features in virtually any instructional explanation (Leinhardt 2001) and examples of all kinds are one of the principle devices used to illustrate and communicate concepts between teachers and learners (e.g. Bruner *et al.* 1956, Tall & Vinner 1981, Peled & Zaslavsky 1997).

Liz, Bills, Tommy Dreyfus, John Mason, Pessia Tsamir, Anne Watson, & Orit Zaslavsky (2006). Exemplification in Mathematics Education. In J. Novotna (Ed.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.* Prague, Czech Republic: PME.

Diagrams, symbols and reasoning are all treated as particular yet thought about (by the teacher at least) as general. Examples offer insight into the nature of mathematics through their use in complex tasks to demonstrate methods, in concept development to indicate relationships, and in explanations and proofs. The core issue is whether learners and teachers are perceiving the same (or indeed any) generality.

An important pedagogic distinction can be made between examples of a concept (triangles, integers divisible by 3, polynomials etc.) and examples of the application of a procedure (finding the area of a triangle, finding if an integer is exactly divisible by 3, finding the roots of a polynomial etc). Sowder (1980) tried to avoid this confusion by distinguishing between ‘examples’ and ‘illustrations’. However, within the category of ‘examples of the application of a procedure’, or ‘illustrations’ we distinguish further between ‘worked(-out) examples’, in which the procedure being applied is performed by the teacher, textbook author or programmer, often with some sort of explanation or commentary, and ‘exercises’, where tasks are set for the learner to complete. The worked-out example has been the subject of a body of research within psychology (e.g. Atkinson *et al.* 2000, Renkl 2002).

Of course, these distinctions are neither precise nor clear cut. Gray & Tall (1994) underline the fact that the same notation may be viewed as signifying a process or an object, so that, for example, a teacher may offer a representation of the function $y = 2x + 3$ as an example of a linear function, but the learner may see it as an example of a procedure (for drawing a graph from an equation). There is a good deal of ‘middle ground’ between exercises and worked examples, for instance when a teacher ‘leads’ a class through the working out of a typical problem using questions and prompts.

Across these broad categories of form and function of examples there are three special descriptive labels: ‘generic example’, ‘counter-example’ and ‘non-example’. Generic examples may be examples of concepts or of procedures, or may form the core of a generic ‘proof’. Counter-examples need a hypothesis or assertion to counter, but they may do this in the context of a concept, a procedure or even (part of) an attempted proof. Non-examples serve to clarify boundaries and might do so equally for a concept, for a case where a procedure may not be applied or fails to produce the desired result or to demonstrate that the conditions on a theorem are ‘sharp’. In fact all three labels have to do with how the person (teacher or learner) perceives the mathematical object in question, rather than with qualities of the object itself.

The term *example* here includes anything used as raw material for generalising, including intuiting relationships and inductive reasoning; illustrating concepts and principles; indicating a larger class; motivating; exposing possible variation and change, etc. and practising technique (Watson & Mason 2002a, 2002b). *Exemplification* is used to describe any situation in which something specific is being offered to represent a general class to which learners’ attention is to be drawn. A key feature of examples is that they are chosen from a range of possibilities (Watson & Mason 2005 p238) and it is vital that learners appreciate that range.

3. EXAMPLES FROM A HISTORICAL PERSPECTIVE

The whole point of giving worked examples is that learners appreciate them as generic, and even internalise them as templates so that they have general tools for solving classes of problems. Unfortunately their use in lessons is often reduced to the mere practice of sequences of actions, in contrast to a more investigative approach (Wallis 1682) in which learners experience the mathematisation of situations as a practice, and with guidance abstract and re-construct general principles themselves.

Whereas mathematical investigations and the use of ‘authentic or ‘modelling’ approaches appear to be a relatively recent pedagogic strategy, there are historical precedents. The earliest mathematical records (Egyptian papyri, Babylonian tablets and later copies of lost Chinese manuscripts) all use context-based problems with worked solutions to illustrate procedures, or what came to be called *rules* and then later *algorithms* in medieval and renaissance texts. They sometimes point specifically to a generality with comments such as ‘thus is it done’ or ‘do it thus’ (Gillings 1972, p. 232), and ‘this way you may solve similar problems’ or ‘by the same method solve all similar problems’ (Treviso Arithmetic 1478, see Swetz 1987, p. 151).

By the 16th century European authors of mathematical texts had begun to justify the presence of examples in their texts, commenting explicitly on the role that examples play for learners. Girolamo Cardano (1545, see Witmer 1968) used phrases such as:

> We have used a variety of examples so that you may understand that the same can be done in other cases and will be able to try them out for the two rules that follow, even though we will there be content with only two examples; It must always be observed as a general rule … ; So let this be an example to you; by this is shown the *modus operandi* in questions of proportion, particularly; in such cases (Witmer 1968, pp. 36-41).

By the late 19th and early 20th century, pedagogic principles become more and more explicit in some cases, if only to attract teachers to ‘new’ pedagogic approaches. For example a textbook from Quebec (MacVicar 1879) claims that:

> The entire drill and discussions [examples] are believed to be so arranged, and so thorough and complete, that by passing through them the pupil cannot fail to acquire such a knowledge of principles and facts, and to receive such mental discipline, as will prepare him properly for the study of higher mathematics. (piv)

Some authors scramble different types of problems, or different looking problems, presumably to engage the learner in recognizing the type, while others collect exercises according to the technique needed, perhaps to promote a sense of the general class of which the exercises are but particulars but more probably to focus on fluency of performance. For example, the expansion by the schoolmaster Iohn Mellis (Record 1632) of John Dee’s extension of Robert Record’s original arithmetic (Record 1543/1969) offers collections of worked examples which offer a variety of differences in what is given and what is sought, so as to draw attention to a wider class of problem type that can be solved by the same method or ‘Rule’.

The design of sequences of examples is a central issue in their instructional use that influences both the inductive and deductive aspects of learning. For example George Pólya (1962) provided long sequences of exercises building up generalisations from a simple starting idea. He ended one such a chapter with a final task:

> Devise some problems similar to, but different from, the problems proposed in this chapter – especially such problems as you can solve. (Pólya 1962, p. 98)

The idea that creating your own examples and questions can aid learning is not new. Record has his scholar in dialogue with the author constructing examples, and Cardano invites the reader to construct their own examples of questions.

Historically there have been two main approaches to the use of examples, distinguished in the 18th century by the terms *analytic* and *synthetic*. The difference amounted to whether general rules were presented before or after worked examples (or even not at all). In the early 19th century Warren Colburn instituted in the USA the inductive method advocated by Johann Pestalozzi (1801):

> The reasoning used in performing these small examples is precisely the same as that used upon large ones. And when anyone finds a difficulty in solving a question, he will remove it much sooner and much more effectively, by taking a very small example of the same kind, and observing how he does it, than by [resorting] to a rule. (Colburn 1826, pp. 141-142)

Herbert Spencer (1878), developed the ideas further, expecting learners to infer the general from carefully presented particulars

> Along with rote-teaching, is declining also the nearly-allied teaching by rules. The particulars first, and then the generalizations, is the new method … which, though 'the reverse of the method usually followed, which consists in giving the pupil the rule first' is yet proved by experience to be the right one. Rule-teaching is now condemned as imparting a merely empirical knowledge – as producing an appearance of understanding without the reality. To give the net product of inquiry without the inquiry that leads to it, is found to be both enervating and inefficient. General truths to be of due and permanent use, must be earned. … While the rule-taught youth is at sea when beyond his rules, the youth instructed in principles solves a new case as readily as an old one. (Spencer 1878, pp. 56–57)

Alfred Whitehead summarised this approach as

> To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought. (Whitehead 1911, p. 4)

Pólya asserted:

> [in doing mathematics]… we need to adopt the inductive attitude [which] requires a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations. (Pólya 1945, p. 7).

Even more important than the distinction between inductive and deductive, between 'general first' or 'general later', are finer distinctions and hybrid approaches which will emerge in later sections. Both inductive and deductive approaches are compatible with constructive accounts of learning and rely on exemplification: inductive learning implies that the learner is making some generalisations about actions or concepts while working with a range of examples (seeing generality through particulars); deductive learning implies that the learner is able to make personal sense of a definition or general principle, and adapt it for current and future use (seeing particular instances in the general).

Examples can be useful stimuli for prompting self-explanation leading to understanding. Cardano acknowledges that sometimes it is too confusing to state a general method, and suggests that examples provide explanation. This sentiment is reflected in a wide range of text authors over the centuries, and by Richard Feynman:

> I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go (Feynman 1985, p. 244).

By contrast, Zazkis (2001) observes that starting with more complex problem situations and more complex numbers not only provides an opportunity for learners to simplify for themselves in order to see what is going on before returning to the more complex, but also provides an opportunity for learners to appreciate more fully the range and scope of generality implied by the particular exemplars. Furthermore, learners are not deceived by the attraction of doing simple computations with small numbers rather than attending to underlying structure.

This survey illustrates a diversity of approaches to examples in learning and teaching. In some cases the succession of examples is the important feature of their use. Their explicit and implicit similarities and differences, the number and variety exhibited, and their increasing complexity can all be used to promote inductive learning. In other cases a single example is intended as a generic placeholder for a completely general expression of a concept, object or process to support deductive thinking.

4. EXAMPLES FROM A THEORETICAL PERSPECTIVE

Examples play a key role in various classes of theories of learning mathematics. Social and psychological forces and situational peculiarities influence and inform both the examples and the concept images to which someone has access at any moment. The notion of a *personal example space* nicely complements the notion of a *concept image* in this respect. Thinking in terms of variation highlights the importance both of the succession of examples and the aspects which are varied in that succession in affording learners access to key features of a concept or technique.

4a The role of examples in doing mathematics

Various mathematicians have written about the importance of examples in appreciating and understanding mathematical ideas and in solving mathematical problems (e.g. Pólya, Hilbert, Halmos, Davis, Feynman). Whenever a mathematician encounters a statement that is not immediately obvious, the 'natural' thing to do is to construct or call upon an example so as to see the general through intimate experience of the particular (Courant 1981). When a conjecture arises, the usual practice is alternately to seek a counter example and to use an example perceived generically to see why the conjecture must be true (Davis & Hersh 1981).

Often a mathematician will detect and express a structural essence which lies behind several apparently different situations. Out of this arises a new unifying concept and an associated collection of definitions and theorems. Sometimes a particular example will suggest some feature which can be changed, leading to a richer or more unifying concept, or at least to an enriched awareness of the class of objects encompassed by a theory. It is not examples as such which are important to mathematicians, but what is done with those examples, how they are probed, generalised, and perceived.

4b The role of examples in theories of learning mathematics

The importance of encounters with examples has been a consistent feature of theories and frameworks for describing the learning of mathematics This section offers a very brief overview of different ways in which theories of learning have used examples.

How people abstract or extract a concept from examples has been specifically studied in psychology from the point of view of how examples and non-examples influence the discernment of concepts (e.g. Bruner 1956, Wilson 1986, 1990, Charles 1980, Petty & Jansson 1987). In Artificial Intelligence attention on default parameters (expectations and assumptions) for triggering frames (patterns of behaviour) were used to try to reproduce concept acquisition (e.g. Winston 1975, Minsky 1975).

Genetic epistemology (Piaget 1970, see also Evans 1973) assumes that individuals actively try to make sense of their world of experience, supported by social groupings (Confrey 1991) in which they find themselves. It underpins many current theories of mathematics learning, by assuming the impact of new examples on existing mental schema through assimilation and accommodation. Piaget’s notion of *reflective abstraction* (Dubinsky 1991) implies experiences and actions performed by the learner through which abstraction is possible.

Building on Piaget’s notion of *schema*, Skemp (1969) wrote about the learning of mathematical concepts through abstraction from examples, which meant that the teachers’ choice of which examples to present to pupils was crucial. His advice on this topic includes consideration of *noise*, that is the conspicuous attributes of the example which are not essential to the concept, and of *non-examples*, which might be used to draw attention to the distinction between essential and non-essential attributes of the concept and hence to refine its boundaries. Once a concept is formed, later examples can be assimilated into that concept (Skemp 1979) and a more sophisticated *concept image* can be formed (Tall and Vinner 1981). Vinner (1983, 1991) describes a gap between learners’ concept image and the concept definition: concept images can be founded on too limited an exploration of the examples encountered so that features of the examples which are not part of the concept are retained in the concept image, a process recognised and elaborated on by Fischbein (1987) as *figural concepts*. Concept images are therefore often limited to domains with which learners are most familiar and so may be too limited to be useful. A considerable part of research results on wrong, alternative and partial conceptions can be convincingly interpreted in this way. Thus improving learners' conceptions amounts to reducing the gap between their concept images and the concept definition. Tall and Vinner point to the importance of the examples in closing this gap.

Thorndike *et al.* (1924) followed a behaviourist line in using examples as stimuli to provoke learning responses, and Gagné (1985) developed this into a hierarchy of behaviours of increasing complexity. Dienes (1960) used cleverly constructed games and structured situations as examples of mathematical structures in which to immerse learners so that they would experience examples of sophisticated mathematical concepts through their own direct experience. Others follow historical precedents in trying to describe what it is like for learners to make sense of new concepts (Davis 1984) and worked examples (Anthony 1994).

Marton and colleagues (Marton & Booth 1997, Marton & Tsui 2004) developed the notion of varied examples as a way to encounter concepts noting that what is needed is variation in a few different aspects closely juxtaposed in time so that the learner is aware of that variation *as* variation. Marton even formulates a definition of learning as becoming aware of one or more dimensions of variation which an example could exhibit. Since teacher and learner may not appreciate the same dimensions of variation, Watson & Mason (2005) expanded this to appreciating a particular concept as being aware of *dimensions of possible variation* and with each dimension, a *range of permissible change* within which an object remains an example of the concept.

Recent articulations which connect the genesis of mathematical knowledge with the processes of coming to know also clarify the central role of examples as the raw material for generalizing processes and conceptualizing new objects. Sfard (1991) follows Freudenthal (1983) in seeing learners moving from an operational to a structural understanding of concepts through a process of *interiorisation* and *condensation* leading to *reification*. Interiorisation and condensation are slow, gradual processes, taking place over time and through repeated encounters with examples. Dubinsky and his colleagues (see Asiala *et al.* 1996) have introduced a theory of the development of mathematical knowledge at undergraduate level which they call APOS theory (actions, processes, objects, schemas). Again the theory predicts that encounters with examples will be part of the process by which learners will move from action to process and then to object conceptions. The Pirie & Kieren (1994) onion model of the growth of understanding focuses on image construction and folding back between states, yet still recognizes that it is direct experience of examples which contribute to the formation of personal images and knowings.

Another aspect of the relationship between examples and concepts or processes centres on the notion of generic example, or prototype. A generic example:

> involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff 1988, p. 219)

Freudenthal (1983) describes examples with this potential as *paradigms*. A strand of psychological research beginning with Rosch (1975) has explored how these *prototypes* (representatives of categories) are used in reasoning. Hershkowitz (1990) drew attention to the tendency to reason from prototypes rather than definitions in mathematics, and the errors that this kind of reasoning can produce. Often learners' concept image is largely determined by a limited number of prototype examples (e.g. Schwarz & Hershkowitz 1999) so it is important to go beyond prototypes using non-typical examples to push toward and beyond the boundary of what is permitted by the definition, becoming aware of that boundary during the process (the range of permissible change). Approaches to helping learners expand their reasoning beyond prototypes have been described in a number of specific areas of mathematics.

Dreyfus (1991) discusses the role of examples in abstraction, and in particular the different uses that might be made by learners of single examples and collections of examples. He suggests that, for a relatively sophisticated mathematical learner, a definition and a single example may be sufficient, whereas less experienced learners may need large numbers of carefully selected examples before they can abstract the properties of the concept.

4c The theory of personal example spaces

The collection of examples to which a learner has access at any moment, and the richness of interconnection between those examples (their accessible *example space*) plays a major role in what sense learners can make of the tasks they are set, the activities they engage in, and how they construe what the teacher-text says and does. Zaslavsky & Peled (1996) point to the possible effects of limited example-spaces accessible to teachers with respect to a binary operation on their ability to generate examples of binary operations that are commutative but not associative or vice versa.

Watson and Mason (2005) formulated the notion of a *personal example space* as a tool for helping learners and teachers become more aware of the potential and limitations of experience with examples. They identify two principles:

> Learning mathematics consists of exploring, rearranging, gaining fluency with and extending your example spaces, and the relationships between and within them.
>
> Experiencing extensions of your example spaces (if sensitively guided) contributes to flexibility of thinking and empowers the appreciation and adoption of new concepts.

A personal example space is what is accessible in response to a particular situation, to particular prompts and propensities. Example spaces are not just lists, but have internal idiosyncratic structure in terms of how the members and classes in the space are interrelated. Their contents and structures are individual and situational; similarly structured spaces can be accessed in different ways, a notable difference being between algebraic and geometric approaches. Example spaces can be explored or extended by searching for situationally-peculiar examples as doorways to new classes; by being given further constraints in order to focus on particular characteristics of examples; by changing a closed response into an open response; by glimpsing the infinity of a class represented by a particular.

4d Summary

While there is a long history of attention to the provision of suitable examples intended to indicate the salient features which make examples exemplary, recent developments indicate that social and psychological forces and peculiarities play a central role in both the personal example space to which learners have access and the concept image which they develop. Particular attention needs to be paid to the succession of examples and both the dimensions of possible variation and their associated ranges of permissible change to which learners are afforded access.

5. EXAMPLES FROM A TEACHER'S PERSPECTIVE[1]

The treatment of examples presents the teacher with a complex challenge, entailing many competing features to be weighed and balanced, especially since the specific choice of and manner working with examples may facilitate or impede learning. Note that the aspects mentioned here are interrelated, not disjoint.

5a. Examples as tools for communication and explanation

Examples are a communication device that is fundamental to explanations and mathematical discourse (Leinhardt 2001). The art of constructing an explanation for teaching is a highly demanding task (Ball 1988; Kinach 2002a, 2002b), as described by Leinhardt *et al.* (1990):

> Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. […]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases. (p. 6, *ibid*).

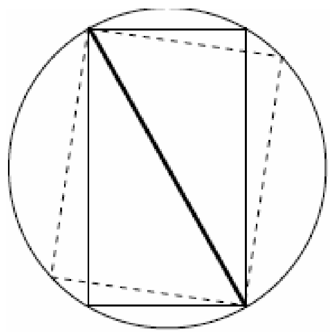
Leinhardt & Schwarz (1997) claim that when teaching meta-skills

> The purpose of an instructional explanation is to teach, specifically to teach in the context of a meaningful question, one deserving an explanation. (p. 399, *ibid*).

That is to say that the meaningful question, the example, plays a key role in the instructional explanation.

Peled & Zaslavsky (1997) distinguish between three types of counterexamples suggested by mathematics teachers, according to their explanatory power: specific, semi-general and general examples. They assert that general (counter)-examples explain and give insight regarding the reason why a specific conjecture is not true and strategies to produce more counterexamples.

The conjecture that two rectangles with the same diagonal must be congruent, is false. The diagram (taken from Peled & Zaslavsky 1997) can be regarded as a general counter example because it communicates an explanation of why the conjecture is false without reference to particular values. Furthermore, inherent to this example is the notion that there are an infinite number of different rectangles with the same diagonal.



With respect to communication, a teacher must take into consideration that an example does not always fulfil its intended purpose (Bills 1996; Bills & Rowland 1999). Mason & Pimm (1984) suggest that a generic example that is meant to demonstrate a general case or principle may be perceived by the learners as a specific instance, overlooking its generality. What an example exemplifies depends on context as well as perceiver.

Attributes which make an example 'useful' include:

Transparency: making it relatively easy to direct the attention of the target audience to the features that make it exemplary.

Generalisability: the scope for generalisation afforded by the example or set of examples, in terms of what is necessary to be an example, and what is arbitrary and changeable.

Examples with some or all of these qualities have the potential of serving as a reference or model example (Rissland-Michener 1978), with which one can reason in other related situations, and can be helpful in clarifying and resolving mathematical subtleties. Clearly, the extent to which an example is transparent or useful is subjective. Thus, the role of the teacher is to offer learning opportunities that involve a large variety of 'useful examples' (yet not too large a variety that might be confusing) to address the diverse needs and characteristics of the learners.

To illustrate some of the distinctions mentioned so far, consider the following examples of a quadratic function (these examples and the subsequent elaboration appear in Zaslavsky & Lavie 2005, submitted):

$$y=(x+1)\cdot(x-3); \qquad y=(x+1)^2-4; \qquad y=x^2-2x-3$$

These are three different representations of the same function. Each example is more transparent about some features of the function and more opaque with respect to others (e.g., roots; position of the vertex and minimum value; *y*-intercept). However, these links are not likely to be obvious to the learner without some guidance on how to read or interpret the expressions. Moreover, it is not even clear that learners will consider all three as acceptable examples of a quadratic function, since, for example the power of two is less obvious in the factored form, and a quadratic may have been defined to look like the third expression. A teacher may choose to deal with only one of the above representations, or s/he may use the three different representations in order to exemplify how algebraic manipulations lead from one to another, or in order to deal with the notion of equivalent expressions.

Each different representation communicates different meanings and affords different mathematical engagement, but there are further possible differences in perception. What a learner will see in each example separately and in the three as a whole depends on the context and classroom activities surrounding these examples, and her own previous experience and disposition. A learner who appreciates the special information entailed in each representation may be informed by them to be alert to their differing qualities in the future, even to the extent of effectively using them as reference examples or reference forms when investigating other (quadratic) functions.

To an expert there are some irrelevant features, such as the use of particular letters yet, a learner may regard x and y as mandatory symbols for representing a quadratic function. Another irrelevant feature is the fact that in all three representations all the numbers are integers. A learner may implicitly consider this to be a relevant feature, unless s/he is exposed to a richer example-space. Learners may also generalise and think that all three representations can be used for any quadratic function.

None of these considerations need be conscious; even the learner who is not *deliberately* making sense of what is offered is still becoming familiar with a particular range of examples which create a sense of normality. Hence, the specific elements and representation of examples, and the respective focus of attention facilitated by the teacher, have bearing on what learners notice, and consequently, on their mathematical understanding. Paul Goldenberg (personal communication) pointed out that sometimes an example can be too specific to be useful; learners and teachers need to be aware that the shift to seeing examples as 'representative and therefore arbitrary' is non-trivial and may need classroom discussion.

5b. Uses of examples for teaching

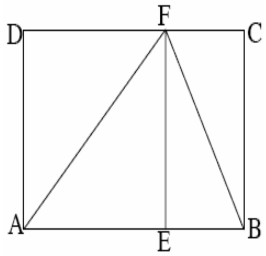
Some authors have categorised examples according to the use for which they are particularly suited. Notable amongst these, Rissland-Michener (1978) distinguishes four types of examples (not necessarily disjoint), which have epistemological significance: start-up (which help motivate basic definitions and results, and set up intuitions in a new subject), reference (which are used as standard instances of a concept or a result, model, and counterexample and referred to repeatedly in the development of theory), model (which are paradigmatic, generic examples) and counterexamples (which demonstrate that a conjecture is false and are used to show the importance of assumptions or conditions in theorems, definitions and techniques).

Rowland and Zaslavsky (2005) distinguish between providing examples of something as raw material for inductive reasoning, as particular instances of a generality, and providing an environment for practice. For example, in order to teach subtraction by decomposition, a teacher might work through say, 62-38 in column format; for practice a collection of well-chosen subtly varying particular cases might be set as an exercise. In the case of concepts, the role of examples is to facilitate abstraction. Once a set of examples has been unified by the formation of a concept, subsequent examples can be assimilated by the concept.

Another kind of use of examples in teaching, more often called ‘exercises’, is illustrative and practice-oriented. For us, exercises are examples, selected from and indicative of a class of possible such examples. Typically, having learned a procedure (e.g. to add 9, to find equivalent fractions, to solve an equation), the learner rehearses it on several such ‘exercise’ examples. This is first in order to assist retention of the procedure by repetition, then later to develop fluency with it (Rowland & Zaslavsky 2005). When the teacher repeatedly demonstrates how to perform on these practice exercises, the learning mechanism that is facilitated may share some characteristics of the learning from worked-out examples (see section 6b).

Hejný (2005) notes that the focus of attention needs to be not only on what can be generalised from one example, but also on a structured set of tasks which may direct learners to find a general or abstract idea. For example, he suggests helping learners in primary grades discover a formula for the area of a triangle, by offering them a rich problem situation, from which a general relationship can be induced.

> Divide a given rectangle ABCD by a segment EF to make two rectangles AEFD and EBCF. These rectangles are divided by diagonals AF and BF into four right-angled triangles. Consider eight shapes: five triangles: AEF, AFD, EBF, BCF, ABF and three rectangles: ABCD, AEFD, EBCF.



> Given the area of two of these shapes, find the areas of all the others;
>
> Given the length of three segments from the following: AE, EB, AB, DF, FC, DC, AD, EF, BC find the areas of all the triangles.
>
> What do you have to know to find the area of triangle ABF?

Most of the studies that deal with sets of example suggest that the specific sequence of examples has an impact on learning. In particular, it is recommended to combine examples and non-examples within a sequence of examples, in order to draw attention to the critical features of the relevant examples. There is an argument for examples to be ‘graded’, so that learners experience success with routine examples before trying more challenging ones. However, it should be noted that sequencing examples from ‘easy’ to ‘difficult’ is not always effective (Tsamir 2003). Exercises designed for fluency are likely to be differently structured to exercises designed to promote or provoke generalisation (Watson & Mason 2006).

Leron (2005) uses the term *generic proof* to refer to what Movshovitz-Hadar (1988) calls a *transparent proof* or *pseudo proof*. Leron illustrates generic proofs with reasoning to justify the fact that every permutation can be decomposed as the product of disjoint cycles. As a simpler example consider the proof that the sum of two odd numbers is an even number. One can use two 'general' odd numbers that are not special in any obvious way, e.g. 137 and 2451, and present a 'proof' for these two numbers, e.g.: 137 + 2451= (136 + 1) + (2452 – 1) = 136 + 2452. This form of presentation can be read generically as justification that the sum of *any* two odd numbers is equal to the sum of two even numbers and hence even. However, learner attention has to be directed appropriately in order to have this effect. The specific choice of examples together with the transparency with respect to the main ideas of the proof both play an important role.

Finally, examples (or exercise examples) can be used for assessment of learners' performance and understanding in a broad sense. The more conventional way would be to present learners with examples of problems or mathematical objects and ask them to follow certain instructions (e.g., solve the problem, compare the objects etc.). In this, the teacher assumes that these examples are cases of a more general class of problems or objects, and considers learners' performance with these examples as a representation of their knowledge. In a way, several researchers use carefully selected examples to investigate learners' schemes (e.g., Dreyfus & Tsamir 2004; Peled & Awawdy-Shahbari 2003). Section 7 elaborates on researchers' use of examples.

Another approach that some teachers (as well as researchers) use for revealing learners' conceptions and ways of thinking is by asking learners to generate their own examples of problems and of objects (e.g. van den Heuvel-Panhuizen *et al.* 1995, Zaslavsky 1997, Hazzan & Zazkis 1999, Watson & Mason 2005).

5c. Teachers' choice of examples

Research on teachers' choice of examples is rather scarce. Ball *et al*. (2005) maintain that a significant kind of mathematical knowledge for teaching involves specific choices of examples, that is, considering what numbers are strategic to use in an example. Similarly, Rowland & Zaslavsky (2005) note that the choice of 62-38 in column format to teach subtraction by decomposition is not a random choice: the digits are all chosen with care because constructing examples is not an arbitrary matter, though there is usually some latitude in the choice of effective examples. The 8 could have been a 9; on the other hand, it could not have been a 2. It could have been a 4, say, but arguably the choice of 4 is pedagogically less effective than 8 or 9, because subtracting 4 from 12 would lead some pupils to engage in finger-counting, distracting them from the procedure they are meant to be learning. Attending to the range of change of digits that is permissible without changing the learners' experience (Watson & Mason 2005) is essential in choosing instructional examples.

Novice teachers' poor choices of examples have been documented by Rowland *et al*. (2003) who considered the way in which student teachers give evidence of their subject knowledge in their teaching of mathematics to primary school children, one aspect being the choice of examples. The authors present instances of choices which, in their words, ‘obscured the role of the variable’ (p. 244): reading a clock face set at half past the hour by using the example of half past six; using as the first example to illustrate the addition of nine by adding 10 and subtracting one, adding nine to nine itself. Often the unintentionally ‘special’ nature of an example can mislead learners.

In selecting instructional examples it is important to take into account learners’ preconceptions and prior experience. In particular, careful construction of examples could enable teachers to identify and help learners cope with the effect of previous knowledge and existing schemes (implicit models) on the construction of new knowledge. Research findings on learning could serve as a rich source for teachers’ selection of effective examples for this purpose. For example, Peled & Awawdy-Shahbari (2003) suggest asking learners to compare carefully selected pairs of decimal or common fractions, in order to identify the implicit models by which they operate. An effective example for decimal fractions would be to ask learners which number is bigger: 2.8 or 2.85. Some learners claim that 2.8 is bigger “because tenths are bigger than hundredths”. Similarly, in comparing $\frac{5}{6}$ and $\frac{3}{5}$, some will say that $\frac{3}{5}$ is larger because fifths are larger than sixths, because they focus on the size of the fractional part and ignore the number of parts. Similarly, the study by Tsamir & Tirosh (1999) regarding learners’ tendencies to address inclusion considerations when dealing with comparisons of infinite sets informed the choice of examples Tsamir and Dreyfus subsequently presented to learners (Tsamir & Dreyfus 2002).

In secondary school the considerations in selecting specific examples seem to be far more complex than in primary school. Zaslavsky and Lavie (2005, submitted) and Zaslavsky and Zodik (in progress) discuss teachers’ considerations underlying their choice of examples. Issues that came up in their study include: the tension between the teachers’ desire to construct ‘real-life’ examples and the mathematics accuracy they feel they are ‘sacrificing’ when doing so; the dual message of randomly selected examples since the randomness may convey the generality of the case, however it may also yield impossibilities or inadequate instances; the visual entailments of examples in geometry, and the ambiguity regarding what visual information may be induced and what should not. A classic instance is that when a ‘general’ triangle is sketched, some learners rely on the relative magnitude of length of its sides, leading to examiners asserting with every diagram ‘not drawn to scale’.

5d. Summary of teacher perspective

The use of examples in the classroom is an essential but complex terrain. It involves careful choices of specific examples which facilitate the directing of attention appropriately so as to explain and to induce generalisations. Desirable choice of examples depends on many factors, such as the teaching goals and teachers’ awareness of their learners’ preconceptions and dispositions.

It has been proposed (*e.g.*, Tall & Vinner 1981, Chi *et al.* 1989, Chapman 1997) that the key feature of learning is not what is presented but rather what is encoded in the learner's mind, what is constructed by the learner, what practices are internalised.

6. EXAMPLES FROM A LEARNER'S PERSPECTIVE

The crucial factors for appreciating and assimilating concepts, and for learning techniques are the form, format and timing of examples encountered, and experience of ways of working with and on examples. When invited to construct their own examples, learners both extend and enrich their personal example space, but also reveal something of the sophistication of their awareness of the concept or technique.

6a. Concept formation

Davis (1984) described mathematical objects emerging from specific experiences:

> When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a thing. […] The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun. (pp. 29-30, *ibid*).

In the process of concept formation, the operational conception (focussing on the process) is often first to develop, gradually moving towards a structural approach (focusing on the object) (Rumelhart 1989). Gray and Tall (1994) use the example '2 + 3' to illustrate how a symbol sequence or expression may be conceived either as aa process (add) or a concept (sum). A learner might perceive an example either as a process, or as an object, or both (*proceptually*). For example, if a learner's only experience of equations is of being shown how to solve them, with the language only of 'doing', then it is unlikely that a conceptual understanding will be formed easily.

Charles (1980) argues that while for 'easy' concepts a sequence of examples from which to generalise may be sufficient, for more 'difficult' concepts non-examples are also necessary to delineate the boundaries of the concept. Wilson (1986) points out that learners can be distracted by irrelevant aspects of examples, so the presence of non-examples provides more information about what is, and is not, included in a definition. Since examples are far more effective than formal definitions in appreciating concept (Vinner 1991), learning might be enhanced by contact with a rich variety of examples and non-examples. Paul Goldenberg (private communication) observed that there is a big difference between noticing for oneself a salient feature in a collection of examples and then naming it, and being given a new word followed by a sequence of objects which are supposed to illustrate its meaning.

How rich and in what variety needs careful study however. Bell (1976) reported that school learners often do not recognize the significance of counterexamples and would not necessarily alter their conjectures or proofs if a counterexample did crop up, and this is reflected in the observation that undergraduates also tend to monster-bar

(MacHale 1980) rather than modify their concept image. It is fairly obvious that a limited experience of examples and non-examples may lead to a restricted concept image, but it is also the case that limiting mathematics to sequences of examples ‘to be done’, rather than sets of examples to be understood, may induce learners to focus on completing their tasks rather than on making sense of the tasks as a whole (Watson & Mason 2006). A succession of examples does not add up to an experience of succession. Not attending to the whole may result in an overly restricted understanding of the nature of mathematics.

6b. Learning from worked-out examples

Several studies point to the contribution of worked-out examples for learning to solve mathematical problems (e. g. Reed *et al*. 1985; Reimann & Schult 1996; Sweller & Cooper 1985). However, providing worked-out examples with no further explanations or other conceptual support is usually insufficient. Learners often regard such examples as specific (restricted) patterns which do not seem applicable to them when solving problems that require a slight deviation from the solution presented in the worked-out examples (Reed *et al.* 1985, Chi *et al*. 1989). Note however that the immensely insightful mathematician Ramanujan was, while a student, able to treat a book of summarised generalities as a sequence of particular examples!

Watson & Mason (2002a, 2002b) suggest that worked-out examples might even inhibit learners' ability to generalise apart from recognition of the syntactical template. One explanation of this phenomena was given by Reimann & Schult (1996), based on Artificial Intelligence literature. They claim that the information captured and attention drawn in worked-out examples is mostly the solution steps, which limit matching and modification processes. Furthermore, Reimann & Schult (*ibid*) assert that it is important to specify in a worked-out example the steps that were taken and the reasons for taking them, that is, how attention is directed. This is consistent with the findings of Chi *et al.* (1989) and Renkl (2002) who emphasise the importance of learners’ self-explanation of the worked-out example, and also with the work of Eley & Cameron (1993) who found that learners considered an explanation to be better if it included the ‘trigger’ for each step. Worked-out examples *may* enhance learners' learning, and in particular their problem solving performance, but only if they are used in ways which encourage explanation and reasoning.

Much of the research in this area has been directed towards a view of learning as measurable by performance of techniques and solution of word problems, rather than of learning as conceptual understanding or mathematical enquiry. The role of worked-out examples in conceptual understanding deserves further research.

6c. The role of examples in mathematical reasoning and problem solving

Examples can play a role in facilitating non-routine problem solving, a process in which reasoning about the situation allows the learner to apply and adapt sequences of techniques whose purposes need to be understood. If this is seen as a process of applying known techniques, the relevant worked-out examples which the learner has experienced need to be sufficiently different, and sufficiently explained, for the purpose of the techniques used to be understood. If, on the other hand, problem-solving is seen as a process of modelling a situation and tackling it heuristically, a learner needs to have some knowledge of similar situations in order to be successful.

One of the main processes of reasoning about novel situations is reasoning by appealing to similarity (Rumelhart 1989). Rumelhart refers to a continuum, moving from ‘remembering’ a suitable example to ‘analogical reasoning’. Another central kind of mathematical reasoning that necessitates generation of examples is proving by refutation. Addressing learners' difficulties in producing and using appropriate counterexamples is another challenge for teachers' use of examples (Zaslavsky & Peled 1996; Zaslavsky & Ron 1998). Pólya (1945, 1962) elaborates on the processes of inductive (example-based) reasoning, generalization, and analogical reasoning, all of which greatly depend on examples.

It seems that all learners who are even only partially engaged try to generalise from sequences of examples, implicitly or explicitly, and that this is done by the natural process of discerning differences and similarities in what is available to be perceived. What they choose to stress and ignore, and what they ‘get from it’ is highly variable. Discerning invariance and variation explains many standard misconceptions in mathematics: learners generalise inappropriately, but in ways which can be seen to be the products of mathematical reasoning, given their experience. Thus learners are always engaged in mathematical reasoning whenever they are exposed to a set of examples of anything, although this may not be recognised or made explicit.

There are many unresolved issues. For example, Hejny (personal communication) questions whether ‘natural’ generalisation is always the same kind of process, or whether it differs according to whether one is encountering a concept, a process, etc..

Novices and expert mathematicians alike depend on experiences with a single rich generic example, or else, as with most novices, numerous examples, in order to get some intuition about the situation and then try to generalise and reason from them. (Bills & Rowland 1999, Zaslavsky & Lavie 2005). This mixture of logical-based reasoning (using deductive mechanisms) and example-based reasoning (Lakatos 1976) characterises mathematical competence at every level.

Weber & Alcock (2004, 2005) documented how undergraduates learning to prove use examples in reasoning and constructing proofs. They recognised that professional mathematicians switch fluently between examples (specific cases) and formal definitions, so they asked how learners make the transition to this fluency, if this shift has not been made explicit for them. They found that example use for such learners is often illustrative and empirical rather than general and deductive. Where their reasoning failed, they were more likely to self-correct errors to do with the individual example than errors to do with the underlying rationality. Alcock & Weber (in press) then distinguished between two learners who used a *referential* approach to proof and a *syntactical* approach. The learner who used referential approach rejected examples as a tool for developing structural understanding and may have needed help in describing examples more formally, to see how doing so might offer the structure for a formal proof. The learner who approached the task of proof construction as if it were solely a manipulative exercise might have benefited from using specific examples to give her work some meaning, but self-generation of appropriate examples is not trivial for learners who are unused to doing so.

6d. The role of learner generated examples in learning

Learning is an activity which requires initiative and intention. Getting learners to construct their own examples proves to be a highly effective strategy for transferring initiative from the teacher to the learner (e.g. Zaslavsky 1995, Niemi 1996, Dahlberg & Housman 1997, Hazzan & Zazkis 1999, Zazkis 2001, Watson & Mason 2005).

The current shift from teacher-centred to learner-centred pedagogical environments in order to foster mathematical classroom discourse, fits with encouraging learners to construct their own examples, which in turn enables teachers to detect the kinds of understandings reflected by learners' examples (*e.g.* Watson & Mason 2005, and as suggested by Zaslavsky 1995). Creation of an example is a complex task that calls upon conceptual links among concepts (Hazzan & Zazkis 1999). Dahlberg & Housman (1997) showed that learners who generated examples as a strategy of learning were more likely to understand new concepts. 'Give an example of …' tasks prove very useful in assessing learners' understanding (Niemi 1996).

When learners have been asked to create their own examples, they experience the discovery, construction or assembly of a space of objects together with their relationships. Whereas Rissland-Michener (1978) saw *example spaces* as canonically objective, construction is often idiosyncratic, combining modifications of conventional and familiar objects to construct new objects, to recognise new relationships, and to enjoy new meanings and personal understandings.

Easily-available *canonical* spaces, such as those teachers and textbooks commonly use, form suitable starting points for further extension, just as in any learning the learner can only start from what is already known, which may be a proper subset of what is relevant. In other words, through construction, learners become aware of dimensions of possible variation and corresponding ranges of permissible change within a dimension, with which they can extend their example spaces.

From a mathematical perspective it may be possible for an expert to see a large potential space of examples, or at least to have past experience of a large space, but what comes to mind in the moment may only be fragments of that potential. Spaces are often dominated by strong images, some of which may be almost universal. What is accessible in one situation may not be so readily accessible in another. The experience of constructing examples for oneself can contribute to increased sensitivity in future, triggering richer example spaces.

6e. Summary of learner perspective

Examples play a crucial role in learning about mathematical concepts, techniques, reasoning, and in the development of mathematical competence. However, learners may not perceive and use examples in the ways intended by teachers or textbooks especially if underlying generalities and reasoning are not made explicit. The relationship between examples, pedagogy and learning is under-researched, but it is known that learners can make inappropriate generalisations from sets of examples, or fail to make any conceptual inferences at all if the focus is only on performance of techniques. The nature and sequence of examples, non-examples and counter-examples has a critical influence of what opportunities learners are afforded, but even more critical are the practices into which learners are inducted for working with and on examples.

The relationship between examples and logical deduction in proof, or analogical reasoning in problem solving, cannot be assumed to be assimilated or even accommodated by learners without explicit support and provocation. It is valuable for learners to create their own examples, since this process requires complex engagement with concepts and mathematical structures

Learners naturally perceive variation and invariance in what they experience, and make generalisations from this activity, developing example spaces whose contents may be triggered in future situations. How these contents are structured and inter-related is the outcome of past experience and with ways of working with examples.

7. EXAMPLES FROM A RESEARCHER'S PERSPECTIVE

From a researcher's perspective the role of examples in mathematics education research concerns choices based instructional design, in research on learning, and the role of case studies, considered as research examples, in theory development in mathematics education. The three points will be illustrated by means of examples from a research project, in which they are prominent without, however, being explicit.

7a. Research-based design

Research findings depend critically on specific properties of examples just as much as teaching and learning. For example, in the study by Dreyfus & Tsamir (2004); and Tsamir & Dreyfus (2002, 2005), which deals with the comparison of the cardinalities of infinite sets, the task set initially was to compare the numbers of elements in the set of natural numbers with the number of elements in the set of perfect squares. Two representations were used: numeric and geometric. In the numeric representation, the sets were represented on three cards:

Card A {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, …}

Card B {1, 4, 9, 16, 25, 36. 49, 64 81, 100, 121, 144, 169, 196, 225, …}

Card M was identical to Card A. The inclusion relationship was highlighted by asking learners to choose and mark the perfect squares on Card M.

Card M {[1] 2, 3, [4] 5, 6, 7, 8, [9] 10, 11, 12, 13, 14, 15, [16] 17, …}

The geometric representation used squares and the correspondence between side length and area (see Tsamir & Dreyfus 2002, for a detailed description).

The examples in this first task were chosen with attention to research findings (Tsamir & Tirosh 1999) regarding learners' tendencies to think in terms of inclusion when presented with a numeric representation of the task, and to identify the one-to-one correspondence in reaction to the geometric representation of the same task. Consequently, learners may be expected to reach contradictory answers.

After several more tasks using either or both representations as well as algebraic correspondence rules between the sets, the task in the third session was to compare the set A of natural numbers to a set V, which was given numerically as {0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, ...} (Tsamir & Dreyfus 2005). Here the algebraic rule of the one-to-one correspondence is not easily apparent, nor is the establishment of such a correspondence geometrically. Without adequate preparation, a learner could thus be expected to use only inclusion considerations.

When designing a sequence of tasks, whether for the purpose of teaching or research, the characteristics of each specific example need to be taken into consideration. These characteristics include the different representations in which the example can be cast, and whether the example triggers certain types of reasoning, such as analogy or cognitive conflict. Whereas in teaching not all examples will usually be determined ahead of time since inspired and creative teaching involves sensitivity to the flow of events and on the spot decisions by the teacher, in research, on the other hand, researchers usually do plan all examples in advance; nevertheless, decisions to add or omit examples in a specific stage of the research may be made on the basis of the analysis of previous stages or exigencies in the moment.

7b. Research on learning

Learners' abstract mathematical constructs usually emerge from their occupation with specific cases, i.e. examples. This becomes particularly clear in the research mentioned above, which analyzes the case of one learner, Ben, addressing the comparison of powers of infinite sets. How and what exactly learners may or may not learn from examples only becomes clear after detailed, careful and controlled observation, and analysis of the observations, by researchers.

As an example, consider what Ben did (not) learn from the two tasks presented above. When presented with the first task, Ben claimed, as expected, that the number of elements in set A was larger than the number of elements in set B, explaining that "set B is actually part and I mean REALLY part of set A", and that "it is easy to notice that the further I go [in set B] the larger the intervals". Over the next two sessions, Ben gained insight into the problematic aspects of using inclusion and correctly solved this and all other tasks presented to him by using one-to-one mappings between infinite sets in numeric/algebraic and geometric representations. He reached what the researchers interpreted as consolidated in-depth constructs allowing him to solve such tasks, and it seems that this was on the basis of a carefully designed sequence of tasks. For example, with respect to the comparison of set A above with the set of natural numbers greater than 2, he explained:

> "The two extra, unmatched elements stand out and trigger the conclusion that here we have infinity and here infinity plus two, which SEEMS larger. Instead of matching numbers at the same ORDINAL place [pause]. I mean, assuming that if for each place *n* there is one and only one element in each the two sets, then they go on hand in hand, corresponding, and extra elements are just in our imagination. The infinite nature makes it possible that no matter which number you chose in one of the sets, at the same ordinal place there is a matching specific number placed in the other set. It cannot be that the numbers in the second set are finished and cannot provide a matching element, because the set is infinite, and this behavior of plus two goes on, like, forever."

In the third session, Ben was asked to compare the sets A and V (see above). This example, which was intended to introduce more challenging tasks, turned out to provide the researchers with insight into the complexity of what had been interpreted as Ben's consolidated knowledge about the comparison of countable infinite sets. For over 20 minutes, Ben assiduously tried to establish, geometrically or algebraically, a one-to-one correspondence between A and V. He even noticed that there is a one-to-one correspondence between set A and the set of differences between successive elements of V. But then he ended up concluding,

> "The differences between successive elements get larger and larger. Wow! REALLY larger. I see. Set V consists of fewer elements. REALLY fewer."

Even insistent questioning by the interviewer did not sway his opinion. The interviewer remarked:

> "You once told me that using inclusion and correspondence leads to contradiction. And then you read that only equivalence correspondence should be used for comparing infinite sets. Right?"

To this, Ben replied that yes, indeed, using inclusion and one-to-one correspondence may lead to a contradiction, and that he had not used inclusion except to prove that there exists no one-to-one correspondence.

Based as it was on careful choice of a sequence of examples, this research has advanced our understanding of the important characteristics of consolidation (Dreyfus & Tsamir 2004). Equally interestingly, the choice of the introductory example to the third session also turned out to have an important, though unplanned role in the research because it led to modification of our conception of consolidation.

Research on learning is necessarily based on examples because all learning is either fundamentally based on examples, or at least strongly supported by examples. The choice of examples thus influences research on learning, and possibly research results. Are such research results reliable? Not quite. An example was found where Ben’s supposedly consolidated knowledge broke down. Without this example, conclusions about Ben's consolidation of knowledge about the comparison of infinite sets would have been exaggerated.

There are two ways researchers can counterbalance this influence of examples: One is to be acutely aware of it, and attempt to analyze it, thus recognizing the influence, and the possible ensuing limitations of any specific piece of research; and the other is to carry out several parallel research studies using different sets of examples, the subject of the next subsection.

7c. Theory building

It is generally agreed that theory building is one of the aims of research. In mathematics education, researchers' theoretical constructs about X (e.g. a specific learning process such as consolidating) tend to emerge from observation of a few, sometimes of a single example of X, combined with theoretical reflection on X. The small number of examples is a necessary limitation, due to the fact that examples are often “large” in the sense that they may require weeks of detailed observations and subsequent painstaking analysis of the observations.

Research on constructing and consolidating knowledge is a case in point. Learners can be given opportunities for constructing knowledge – but they cannot be forced to construct; researchers thus provide learners with opportunities, and hope they can observe what they are looking for. Consolidating recently constructed knowledge, by definition, is an ongoing process that may last hours or years. Dreyfus & Tsamir (2004) have proposed characteristics of consolidation on the basis of a single, albeit detailed and very carefully analyzed, but still only a single example, namely the example of Ben constructing and consolidating his knowledge about the comparison of infinite sets.

In a similar vein, the entire ‘RBC theory’ made up of *Recognizing*, *Building-with* and *Constructing* (Hershkowitz *et al.* 2001), within which the consolidation research is located, has been proposed on the basis of a single example, a 9th grade learner learning about rate of change as a function. Again, one example has served to propose an entire theory. Subsequently, the same and other researchers have shown that the theory is applicable to many other contexts, possibly after suitable modification. The theory has thus been strengthened and validated. It is important to stress that this validation is based on examples as well. In this sense, examples play a central and crucial role in the establishment of theory, the other basic element of theory building being theoretical reflection.

7d. Summary of research perspective

The choice of examples, and their sequencing, is crucial in instruction. Examples may be chosen for using specific representations and they may be sequenced to go from easy to difficult for triggering analogy, or from difficult to easy for triggering cognitive conflict (Tsamir 2003). Consequently, research on learning mathematics is necessarily based on examples as well, and the choice of mathematical examples may influence research results. Researchers can counterbalance this influence by being aware of it, by taking it into account when drawing conclusions, and by carrying out parallel research studies using different sets of examples.

Moreover, there is a second level of example use in research. A research study, such as the one about Ben, may itself serve as an example that forms the basis for theory building. Additional examples of research studies are a tool for validating the theory.

8. FOR FURTHER RESEARCH

Particular attention needs to be paid to

the sequencing and timing of a succession of examples, and both the dimensions of possible variation and their associated ranges of permissible change to which learners are afforded access.

ways of directing learner attention so as to perceive exemplariness;

ways of drawing teachers' attention to the importance of the choices of examples they make when working with learners;

the role of worked-out examples in concept formation;

ways of directing learner attention so that sets of exercises are pedagogically effective.

REFERENCES

Alcock, L.J. & Weber, K. (in press). Referential and syntactic approaches to proof: Case studies from a transition-to-proof course. *Research in Collegiate Mathematics Education*.

Anthony, G. (1994). The role of the worked example in learning mathematics. In A. Jones *et al*. (Eds.) *SAME papers* (pp. 129-143), Hamilton, New Zealand: University of Waikato.

Asiala, M. Brown, A. DeVries, D. Dubinsky, E. Mathews D. & Thomas K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education*, *6*, 1-32.

Atkinson, R. Derry, S. Renkl, A. & Wortham, D. (2000). Learning from examples: Instructional principles from the worked examples research. *Review of Educational Research*, *70*(2), 181-214

Balacheff, N. (1988). Aspects of Proof in Pupils' Practice of School Mathematics. In D. Pimm, (Ed.), *Mathematics, Teachers and Children* (pp. 216-235). London: Hodder and Stoughton.

Ball, D. (1988). *Unlearning to teach mathematics. Issue paper 88-1*. East Lansing, MI: National Center for Research on Teacher Education.

Ball, D., Bass, H., Sleep, L. & Thames, M. (2005). *A theory of mathematical knowledge for Teaching*. Work-Session presented at the 15th ICMI study conference: The Professional Education and Development of Teachers of Mathematics. Águas de Lindóia, Brazil

Bell, A. (1976). *The learning of general mathematical strategies*. Unpublished Ph.D. thesis. Nottingham, UK: University of Nottingham.

Bills, L. & Rowland, T. (1999). Examples, Generalisation and Proof. In L. Brown (Ed.) *Making meaning in mathematics, Advances in Mathematics Education 1* (pp. 103-116). York, UK: QED.

Bills, L. (1996). The Use of examples in the teaching and learning of mathematics. In Puig L. & Gutierrez A. (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 81–88). Valencia, Spain: PME.

Bruner, J., Goodnow, J., & Austin, A. (1956). *A study of thinking*. New York: Wiley.

Chapman, O. (1997). Metaphors in the Teaching of Mathematical Problem Solving. *Educational Studies in Mathematics, 32*, 201-228.

Charles, R. (1980). Exemplification and characterization moves in the classroom teaching of geometry Concepts. *Journal for Research in Mathematics Education, 11*(1), 10-21.

Chi, M. T. H., Bassok, M. W., Lewis, P., Reiman, P. & Glaser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science*, *13*, 145-182.

Colburn, W. (1826). *Intellectual arithmetic: Upon the inductive method of instruction*. Boston, USA: Reynolds & Co.

Confrey, J. (1991). Steering a course between Vygotsky and Piaget. *Educational Researcher, 20*(2), 29-32.

Courant, R. (1981). Reminiscences from Hilbert's Gottingen. *Mathematical Intelligencer, 3*(4), 154–164.

Dahlberg, R. & Housman, D. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*, 283-299.

Davis, P. & Hersh, R. (1981). *The Mathematical Experience*. Brighton UK: Harvester.

Davis, R. (1984). *Learning mathematics: the cognitive science approach to mathematics education*. Norwood, NJ, USA: Ablex.

Dienes, Z. (1960). *Building up Mathematics*. London, UK: Hutchinson Educational.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*, 271-300.

Dubinsky, E. (1991). Reflective abstraction in mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht, The Netherlands: Kluwer.

Eley, M., & Cameron, N. (1993). Proficiency in the explanation of procedures: A test of the intuitive understanding of teachers of undergraduate mathematics. *Higher Education, 26,* 355-386.

Evans, R. (1973). *Jean Piaget: The man and his ideas*. New York, USA: Dutton.

Feynman, R. (1985). *"Surely you're joking, Mr Feynman!": Adventures of a curious character*. New York, USA: Norton.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht, The Netherlands: Reidel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.

Gagné, R. (1985). *The conditions of learning* (4th edition). New York: Holt, Rinehart and Winston.

Gillings, R. (1972). *Mathematics in the time of the Pharaohs*. Reprinted 1982. New York, USA: Dover.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116–140.

Hazzan, O., & Zazkis, R. (1999). A perspective on "give and example" tasks as opportunities to construct links among mathematical concepts. *Focus on Learning Problems in Mathematics, 21*(4), 1-14.

Hejný, M. (2005). *Examples, abstraction & generalization*. Notes for the mini-conference on Exemplification in Mathematics. Oxford University, June 2005.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher, & J. Kilpatrick (Eds.), *Mathematics and cognition*. Cambridge, UK: Cambridge University Press.

Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32,* 195-222.

Kinach, B. M. (2002a). Understanding and learning to explain by representing mathematics: Epistemological dilemmas facing teacher educators in the secondary mathematics "methods" course. *Journal of Mathematics Teacher Education, 5*, 153-186.

Kinach, B. M. (2002b). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education, 18,* 51-71

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4th edition, pp. 333-357). Washington DC, USA: American Educational Research Association.

Leinhardt, G., & Schwarz, B. (1997). Seeing the problem: An explanation from Pólya. *Cognition and Instruction, 15*(3), 395-434.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Leron, U. (2005). Notes for the mini-conference on Exemplification in Mathematics, Oxford University, June 2005.

MacHale, D. (1980). The predictability of counterexamples. *American Mathematical Monthly, 87,* p. 752.

MacVicar, D. (1879). *A complete arithmetic, oral and written: designed for the use of common and high schools and collegiate institutes.* Montreal, Canada: Dawson Bros.

Marton, F., & Booth, S. (1997). *Learning and awareness.* Hillsdale, USA: Lawrence Erlbaum.

Marton, F., & Tusi, A. (Eds.) (2004). *Classroom discourse and space for learning.* Marwah, NJ, USA: Lawrence Erlbaum Associates.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15,* 227-289.

Minsky, M. (1975). A framework for representing knowledge. In P. Winston (Ed.), *The psychology of computer vision* (pp. 211-280). New York, USA: McGraw Hill.

Movshovitz-Hadar, N. (1988). Stimulating presentations of theorems followed by responsive proofs. *For the Learning of Mathematics, 8*(2), 12-30.

Niemi, D. (1996). Assessing conceptual understanding in mathematics: Representation, problem solution, justification, and explanation. *The Journal of Educational Research, 89*(6), 351-363.

Peled, I., & Awawdy-Shahbari, J. (2003). Improving decimal number conception by transfer from fractions to decimals. *Proceedings of the 27th International Conference for the Psychology of Mathematics Education (*Vol. 4, pp. 1-6). Honolulu, USA: PME.

Peled, I., & Zaslavsky, O. (1997). Counter-examples that (only) prove and counter-examples that (also) explain. *FOCUS on Learning Problems in mathematics, 19*(3), 49-61.

Pestalozzi, J. (1801). (Ed. E. Cooke, Trans. L. Holland & F. Turner 1894). *How Gertrude teaches her children: An attempt to help mothers to teach their own children and an account of the method.* London: Sonnenschen.

Petty, O. S., & Jansson, L. C. (1987). Sequencing examples and non-examples to facilitate concept attainment. *Journal for Research in Mathematics Education, 18*(2), 112-125.

Piaget, J. (1970). *Genetic epistemology*. New York, USA: Norton.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26,* 165-190.

Pólya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton, USA: Princeton University Press.

Pólya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined edition). New York, USA: Wiley.

Record, R. (1543/1969) *The Ground of Arts: teaching the perfect worke and practise of arithmeticke, both in whole numbers and fractions.* London: Harper, Thomas. (Reprinted 1969) Amsterdam, The Netherlands: Da Capo Press.

Record, R. (1632) *The Ground of Arts: teaching the perfect worke and practise of arithmeticke, both in whole numbers and fractions.* London: Harper, Thomas.

Reed, S., Dempster, A., & Ettinger, M. (1985). Usefulness of analogous solutions for solving algebra word problems. *Journal of Experimental Psychology: Learning, Memory and Cognition, 11,* 106-125.

Reimann P., & Schult, T. (1996). Turning examples into cases: Acquiring knowledge structures for analogical problem-solving. *Educational Psychologist*, *31*(2), 123-140.

Renkl, A. (2002) Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction, 12,* 529–556

Rissland, E. (1991). Example-based reasoning. In J. F. Voss, D. N. Parkins, & J. W. Segal (Eds.), *Informal reasoning in education* (pp. 187-208). Hillsdale, NJ, USA: Lawrence Erlbaum Associates.

Rissland-Michener, E. (1978). Understanding understanding mathematics. *Cognitive Science, 2,* 361-383.

Rosch, E. (1975). Cognitive representations of semantic categories. *Journal of Experimental Psychology: General*, *104*, 192-322.

Rowland, T., & Zaslavsky, O. (2005). *Pedagogical Example-Spaces*. Notes for the mini-conference on Exemplification in Mathematics, Oxford University, June 2005.

Rowland, T., Thwaites, A., & Huckstep, P. (2003). Novices' choice of examples in the teaching of elementary mathematics. In A. Rogerson (Ed.), *Proceedings of the International Conference on the Decidable and the Undecidable in Mathematics Education (*pp. 242-245). Brno, Czech Republic.

Rumelhart, D. (1989). Toward a microstructural account of human reasoning. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 298-312). Cambridge, UK: Cambridge University Press.

Schwarz, B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? *Journal for Research in Mathematics Education, 30*(4), 362-389.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Skemp, R. (1969). *The psychology of learning mathematics*. Harmondsworth, UK: Penguin.

Skemp, R. R. (1979). *Intelligence, learning and action*. Chichester: Wiley.

Sowder, L. (1980). Concept and principle learning. In R. Shumway (Ed.), *Research in Mathematics Education* (pp. 244-285). Reston, VA, USA: NCTM.

Spencer, H. (1878). *Education: intellectual, moral and physical*. London: Williams & Norgate.

Sweller, J., & Cooper, G. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition & Instruction, 2,* 58-89.

Swetz, F. (1987). *Capitalism and arithmetic: The New Math of the 15th century.* (Trans. Smith). LaSalle, USA: Open Court.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Thorndike, E., Cobb, M., Orleans, J., Symonds, P., Wald, E., & Woodyard, E. (1924). *The psychology of algebra*. New York, USA: Macmillan.

Tsamir, P. (2003). From "easy" to "difficult" or vice versa: The case of infinite sets. *Focus on Learning Problems in Mathematics, 25,* 1-16.

Tsamir, P., & Dreyfus, T. (2002). Comparing infinite sets – a process of abstraction: The case of Ben. *Journal of Mathematical Behavior, 21,* 1-23.

Tsamir, P., & Dreyfus, T. (2005). How fragile is consolidated knowledge? Ben's comparisons of infinite sets. *Journal of Mathematical Behavior, 24,* 15-38.

Tsamir, P., & Tirosh, D. (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education, 30,* 213-219.

Van den Heuvel-Panhuizen, M., Middleton, J., & Streefland, L. (1995). Student-Generated problems: Easy and difficult problems on percentage. *For The Learning of Mathematics, 15*(3), 21-27.

Vinner, S. (1983). Concept image, concept definition and the notion of function. *International Journal of Mathematics Education in Science and Technology, 14*(3), 293-305.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.), *Advanced mathematical thinking.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Wallis, J. (1682). *Treatise of algebra both historical and practical shewing the original progress, and advancement thereof, from time to time; and by what steps it hath attained to the heighth at which it now is*. London, UK: Richard Davis.

Watson, A., & Mason, J. (2002a). Extending example spaces as a learning/teaching strategy in mathematics. In A. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 377-385). Norwich, UK: PME.

Watson, A., & Mason, J. (2002b). Student-generated examples in the learning of mathematics. *Canadian Journal of Science, Mathematics and Technology Education, 2*(2), 237-249.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ, USA: Erlbaum.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematics Thinking and Learning, 8*(2), 91-111.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics, 25*(1), 34-38.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56,* 209-234.

Whitehead, A. (1911). *An introduction to mathematics* (reprinted 1948). Oxford, UK: Oxford University Press.

Wilson, S. (1986). Feature frequency and the use of negative instances in a geometric task. *Journal for Research in Mathematics Education, 17*(2), 130-139.

Wilson, S. (1990). Inconsistent ideas related to definitions and examples. *Focus on Learning Problems in Mathematics, 12,* 31-47.

Winston, P. H. (1975). Learning structural descriptions from examples. In P. Winston (Ed.), *The psychology of computer vision*. New York, USA: McGraw-Hill.

Witmer, T. (Trans.) (1968). *Ars Magna or the Rules of Algebra, Girolamo Cardano*. New York, USA: Dover.

Zaslavsky, O. & Lavie, O. (2005). *Teachers' use of instructional examples*. Paper presented at the 15th ICMI study conference: The Professional Education and Development of Teachers of Mathematics. Águas de Lindóia, Brazil.

Zaslavsky, O. & Lavie, O. (submitted). *What is entailed in choosing an instructional example?*.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. *Journal for Research in Mathematics Education, 27*(1), 67-78.

Zaslavsky, O., & Ron, G. (1998). Students' understanding of the role of counter-examples. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 225-232). Stellenbosch, South Africa: PME.

Zaslavsky, O. & Zodik, I. (in progress). *The merits and dangers of random selection of examples for mathematics teaching.*

Zaslavsky, O. (1995). Open-ended tasks as a trigger for mathematics teachers' professional development. *For the Learning of Mathematics, 15*(3), 15-20.

Zaslavsky, O. (1997). Conceptual obstacles in the learning of quadratic functions. *FOCUS on Learning Problems in Mathematics, 19*(1), 20-44.

Zazkis, R. (2001). From arithmetic to algebra via big numbers. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* (pp. 676-681). Melbourne, Australia: University of Melbourne.

---

[1] **Parts of this paper, particularly Section 5 - "Examples from a Teacher's Perspective", are based on research supported by the Israel Science Foundation (grant No. 834/04).**