Footholds for Inquiry Oriented Instruction

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NCTM Standards

NCTM Standards, aiming for students to

- become better problem solvers,
- learn to reason mathematically,
- learn to value mathematics,
- become more confident in their mathematical ability, and
- learn to communicate mathematically"

(Maccini and Gagnon, 2002)

Pendulum swings

- Pendulum swings from practicing skills to problem solving & sense making, and back
- Wil it be different this time?

- The Standards are ambitious and hard to enact
- Three decades of research and development; Standards-based, Standards-inspired, problem based, ……. Inquiry Oriented Instruction

Exemplary innovaton

Schoenfeld (2003):

- The Philadelphia Standards-inspired reform (--- # schools) satisfactory results, both
  - understanding, problem solving
  - procedural skills
- Requirements

1. The curriculum materials must support inquiry-oriented instruction
2. Teachers must be able to teach in this manner.
3. Mandated assessment must be in line with the goals of Standards-inspired reform

We know how to do it

Requirements - footholds

1. *Curriculum materials*
   ⇔ instructional design theory: RME
2. *Teaching*
   ⇔ theaching theory: socio-constructivism, IOI
3. *Assessment*
   ⇔ goals, assessment, competencies, KOM

Overview

1. Instructional design theory:
   1. History of RME
   2. Socio-constructivist elaboration of RME
2. Theories on teaching: socio-constructivism, IOI

(3. Goals, assessment, competencies, KOM)

RME, History/background

RME, History/background

RME theory originated from Freudenthal's (1973) philosophy of mathematics education

- Mahematics as a human activity
  - looking for problems
  - solving problems
  - organizing subject matter
    - matter from reality
    - mathematical matter

This can be matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or of others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach. (Freudenthal, 1971, 413-414).

RME, History/background

- Task designers/researchers ‘Freudenthal Institute’, to design instructional sequences that were in line with Freudenthal’s philosophy of mathematics
- Treffers (1987) construed RME theory by *reconstructing* the principles that underpinned this design work

- Treffers (1987)
  - *horizontal mathematization*
    organizing subject matter from reality
  - *vertical mathematization*
    organizing mathematical matter

9

RME, History/background

Treffers’ RME theory was published at a favorable time ⇔ push for understanding and applications & RME theory was connected to a series of prototypical instructional sequences
→ collaborative projects in several countries

US projects fostered a merger of RME and socio-constructivist thinking:
- research projects led by Paul Cobb (----)
- Mathematics in the City project (---)

10

RME & socio-constructivism

- RME practitioners/theorists started to embrace socio-constructivist ideas in varying degrees
- Socio-constructivist elaboration of RME ⇔ Cobb, Yackel and Gravemeijer → ‘inquiry approach’ & socio-constructivist elaboration of RME (>10 year collaboration)
- Rasmussen / RUME group, “Inquiry Oriented Instruction” (build on RME)

11

RME & socio-constructivism

- Classroom culture (New for RME)
- Symbolizing & modeling (≠ socio-constructivists)
  → emergent modeling

- RME revisited; Socio-constructivist elaboration of RME Theory, “*prescriptive*” (instructural design)
- Treffers’ RME theory, *descriptive*
  ⇔ categorizing textbooks:
  - mechanistic, empiristic, structuralistic, realistic

12

Back to our objective, footholds

We know how to do it

1. *I*nstructional design theory: RME
2. *Theories on teaching*: socio-constructivism, IOI
3. *(Assessment)*

RME revisited

Socio-constructivist elaboration of RME Theory; istructional design theory

- Instructional design heuristics:
  - Guided reinvention
  - Didactical Phenomenoloy
  - Emergent Modeling

Guided Reinvention

Freudenthal:

Mathematics as an activity

Students should be supported in reinventing mathematics

- Sources if inspiration:
  - history of mathematics
  - informal solution procedures (Streefland)

Guided Reinvention

- History of mathematics
- Egyptian fractions
- Decimals
- Calculus

- Informal solution procedures
- Fair sharing

Guided reinvention of decimals

T: How would one have measured in old times?
…. Iron wire (of 1 meter)
Task: Mesuring with a rope of 1 meter
- 1 rope plus 4/5 rope
- 1 4/5 ‘rope-length’

T: How to measure more precise?
Ss: in fouths, in siths, in fifths
T: Wat would be a convenient number?
T: Historically: in 10 pieces.

17

Guided reinvention of decimals

Task: Measuring with a paper strip with subsivision in 10ths.
- postcard
  - Students: 1/3 strip

T: What subdivision woud be convenient?
Ss: 6, 12, 28, 10.
T: Historically: 10ths.

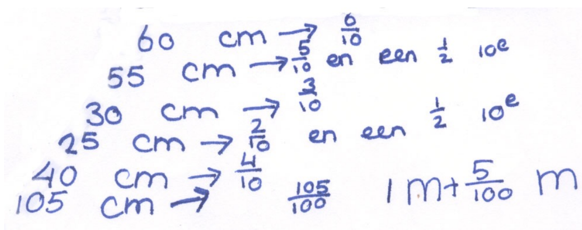
18

Les 3

T: The long strip is 1 meter; the small strip is 10 centimeter.
T: What would 1/10 of the small strip be?
Ss: 1/100 meter.
- Express 60 cm, 55 cm, 30 cm, 25 cm in strip lengths

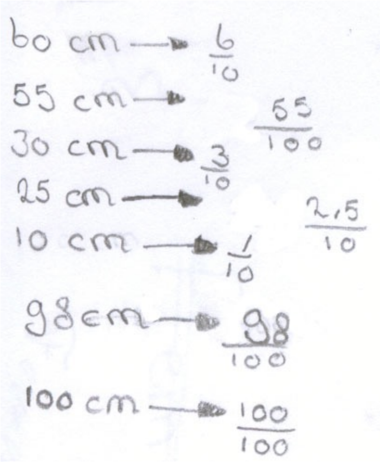
19

Student work

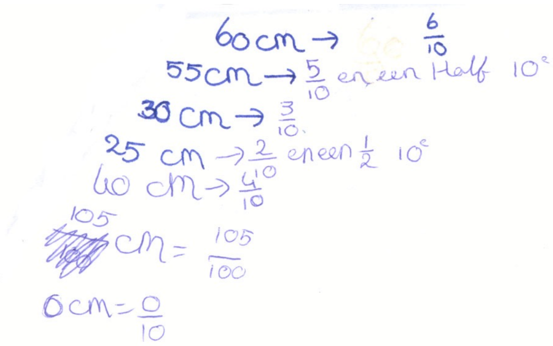


20

Student work

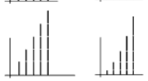


Student work

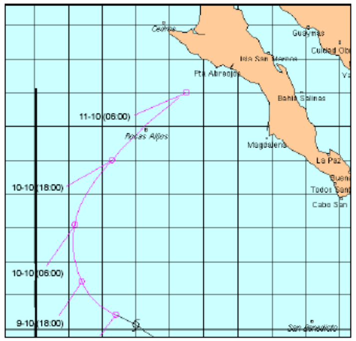


calculus

Table 1 Summary of an emerging instruction theory

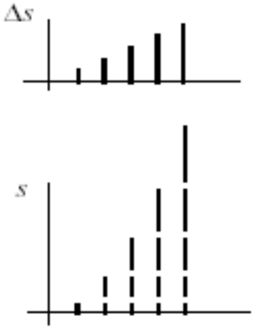
Tool	Imagery	Activity	Concepts
Time series (e.g., satellite photos & stroboscopic pictures) 	Real world representations signify real world situations	Predicting motion (e.g., in the context of weather predictions) Should result in a feeling that the ability to predict motion with discrete data is an important issue	Displacements in equal time intervals as an aid for describing and predicting change
Trace graphs of successive locations 	Signifies a series of successive displacements in equal time intervals	Compare, look for patterns in displacements and make predictions by extrapolating these patterns Resulting in a willingness to find other ways to display displacements for viewing and extrapolating patterns in them	Displacements as a measure of speed, of changing positions, but difficult to extrapolate
Discrete 2-dim graphs 	Signifies patterns in displacements of trace graphs (and cumulative)	Compare patterns and use graphs for reasoning and making predictions about motion (also at certain moments: interpolate graphs) refine your measurements for a better prediction: displacements decrease Should result in the need to know more about the relation between sums and differences, and in the need to know how to determine and depict velocity	Displacements depicting patterns in motion; linear line of summit in graph of displacements or graph of distances traveled; problems with predictions of instantaneous velocity

Calculus & kinematica



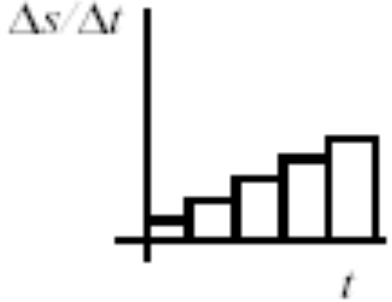
- Discrete benadering van beweging: verplaatsingen per tijdseenheid

Calculus & kinematica



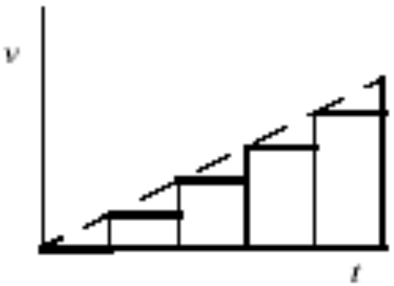
- (Cumulatieve) verplaatsingsgrafiek: samenhang tussen verplaatsing en afgelegde weg

Calculus & kinematica



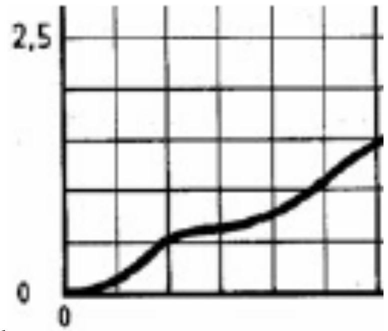
- Continue tijd-as, Δs/Δt als snelheidsmaat
- Oppervlakte als maat voor de afgelegde weg

Calculus & kinematica



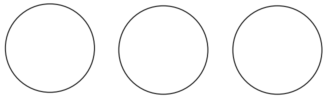
- Benadering met lokaal constante snelheden
- Oppervlakte als maat voor de afgelegde weg

Calculus & kinematica



- Continue grafiek afgelegde weg; discreet benaderen

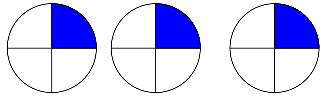
Fractions
Four children share three pizzas



29

Fractions
Four children share three pizzas

- Possible solutions

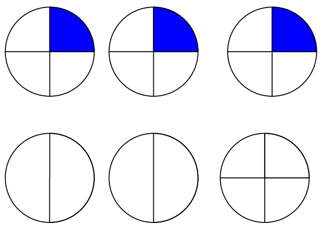


3 x ¼

30

Fractions
Four children share three pizzas

- Possible solutions

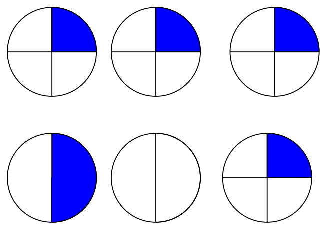


3 x ¼

31

Fractions
Four children share three pizzas

- Possible solutions



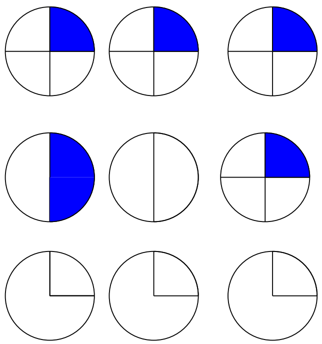
3 x ¼

½ + ¼

32

Fractions
Four children share three pizzas

- Possible solutions



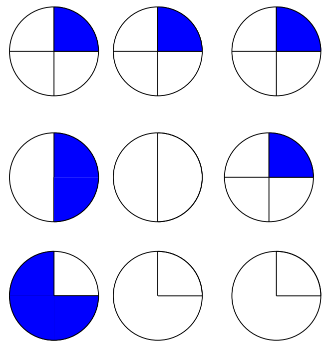
3 x ¼

½ + ¼

33

Fractions
Four children share three pizzas

- Possible solutions



3 x ¼

½ + ¼

¾ = 1 – ¼

34

Guided Reinvention

Starting points should be *experientially real*

- Situations in which students can act and reason sensibly
  - some (!) real-life situations ≠ authentic
  - stories
  - Mathematics

  Contexts are not added to motivate students (sugar coat math), but to create a basis for understanding

Didactical Phenomenology

Freudenthal (1986)

- In the history of mathematics, *mathematical thought things* (such as concepts, rules and procedures) are constructed as means for organizing subject matter
- instructional designers should go in search of, “phenomena that beg to be organized by the thought things that are to be constructed by the students”
- Create a situation
- “What are the phenomena that are organized by this concept, rule or procedure?”

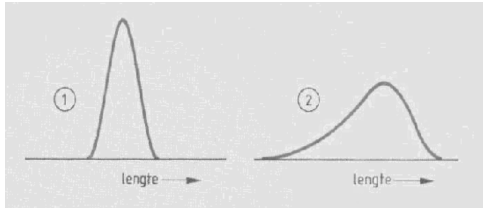
Didactical Phenomenology

- Example "distribution"
- 7th grade: mean, mode, quartiles, extremes, ..
- Tools to get a handle on the distribution
- Alternative goal: distribution as a mathematical object

37

Didactical Phenomenology

- Area ⇔ probability/density distribution
- Graph of a density function
  - height = density of data points around that value
- Distribution can be thought of in terms of shape and density
  - Spread
  - Skewness
  - Position



38

Didactical Phenomenology

What are the phenomena that are organized by this concept, rule or procedure?

(gets you a handle on)

- Density function organizes density of data points
- Density organizes how the data points are distributed
- Data points on an axis organize measurement values

Didactical phenomenology

Instructional design:

Bottom-up: organizing:

- Organizing measurement values
  → data points on an axis
- Organizing the distribution in of data points
  → density
- Organizing density
  → density function

Didactical Phenomenology

- A broad phenomenological exploration (Treffers), which may be exploited to create many inroads for a give topic
- fractions:
  Partiting, measuring, deviding, ratio, proportions,

41

Emergent Modeling

- Friction between the role of models in RME and the constructivist aversion towards models and symbols [classroom observations]

- Learning paradox (Bereiter, 1985):
- "How is it possible to learn the symbolizations you need to come to grips with new mathematics, if you have to have mastered this new mathematics to be able to understand these symbolizations?"
- Circumvent the learning paradox
- History: symbolizing and the development of meaning co-evolve (reflexive relation symbolizing & meaning)(Meira)

42

Emergent Modeling

the emergent-modeling design heuristic,

- A model that first comes to the fore as a *model of* informal mathematical activity and gradually develops into a *model for* more formal mathematical reasoning.

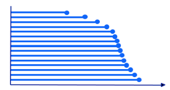
  - overarching model; model-of/model-for
  - sub-models/chain of signification (material correlates of the overarching model)
  - new reality; objects as junctions in a framework of mathematical relations

43

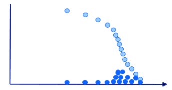
Data Analysis as an example

- Model shift: model of measures/dat values
- Model for reasoning about distribution
- Developing a network of mathematical/statistical relations: notions of density, shape, spread, skewness; omplicit notions of measure and variable
- Series of sub-models:

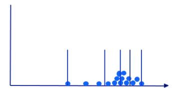
  Value bars



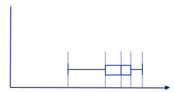
  Dot plot



  Four equal groups



  Box plot



44

Emergent Modeling

Mathematical objects

- Sfard (1991): history of mathematics:

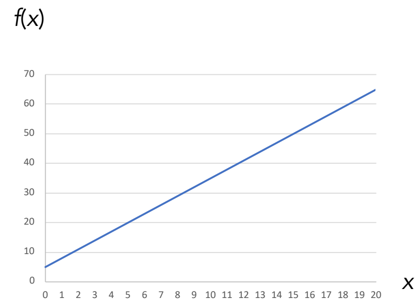
  processes are repeatedly transformed into objects, which in turn become subject to new processes

  Number: counting := natural number
  Fractions: dividing := fractions
  Functions:
  executing calculation prescriptions := functions

45

Proces -Object

- f(x) = 2x + 3 ⇔ ***proces*** (prescription):
  multiply by 2 and add 3
  e.g. x = 4 → '2 × 4 = 8'; '8 + 3 =11'
- Looking at all input values 'x' and corrsponding values



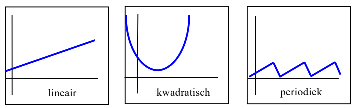
Set of orderd number pairs ⇔ ***object***

46

Reification

Function as an example

- First, prescriptions: how an output value is produced for a given input value. (process).
- Later on, function as a whole: set of ordered number pairs → functions become objects with certain characteristics; such as being linear, quadratic, or periodic.



- Duality: process & object ≈ procept (Gray & Tall)
- shuttle back and forth

47

Teaching

48

Teaching

Socio-constructivism

- Students are expected to
  - explain and justify their thinking;
  - try to understand other students' reasoning,
  - and to ask questions if they don't;
  - challenge arguments, they do not agree with
- This is not what students do in school mathematics
- coffee…..
- Social norms/didactical contract

Emergent perspective (Cobb & Yackel)

Socio-constructivism, History/background

Social perspective

- Classroom social norms
- Socio-mathematical norms
- Mathematical practices

Psychological perspective

- Beliefs about one's own role, the role of others
- Mathematical beliefs and values
- Mathematical interpretations and reasoning

Socio-constructivism, History/background

Sociomathematical norms ⇔ what mathematics is:

- what counts as a mathematical problem
- what counts as a mathematical solution
- what counts as a more sophisticated solution

provide a basis for intellectual autonomy

Socio-constructivism, History/background

Classroom Mathematical Practices

- Who's learning trajectory?
  - All students???
  - Individual learning routes???
- Alternative: sequence of *mathematical practices*: all students are more or less on the same track
  - Initially students have to explain and justify,
  - Later they don't ask for justification anymore⇔ mathematical practice

Socio-constructivism, History/background

How to change the classroom social norms

- Establishing social norms ⇔ experience
  - What is valued
  - What is rewarded
- Using instances as opportunities to clarify norms

- Mr. K.: "How many?"
- Donna: "Eight"
- Mr. K.: "How many?"
- Donna: "Eh, … seven(?)"

Next Mr. K. moves to other students. Later as it is Established that 8 was the right answer, Donna complains

- Donna: "I said eight but you said I was wrong!"
- Mr. K.: "What is your name?"
- Donna: "Dona"
- Mr. K.: "What is your name?"
- Donna: "Dona"
- Mr. K.: "And if I would ask you again, "What is your name?", would you say anything else but Donna?"

Teaching

Simon (1995)

"If the students construct their own mathematics, how do I make them construct, what I want them to construct?

hypothetical learning trajectory

- envision the mental activities of the students
⇔ goals of the lesson

hypothetical: check expectations

→ anticipate, enact, analyze, reflect, revise

Example

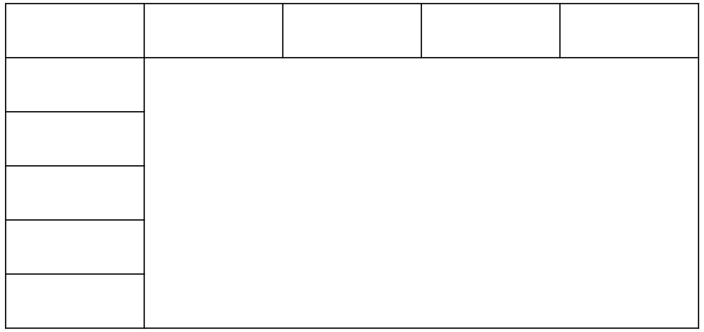
Do my students (student-teachers) understand area.

- Area = length x width
- Blind algorithm??

Rectangles problem 1.

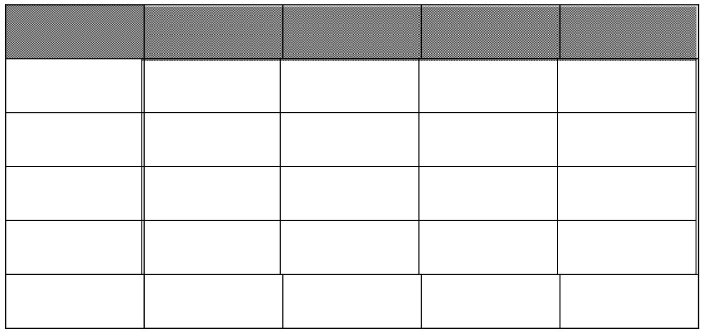
Determine how many cardboard rectanglesfit on the top surface of your table

57



58

59

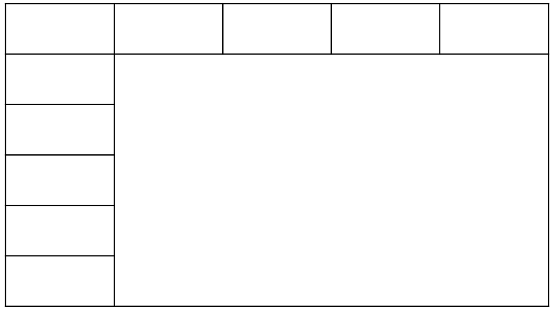


60

Rectangles problem 2.

Bill said,

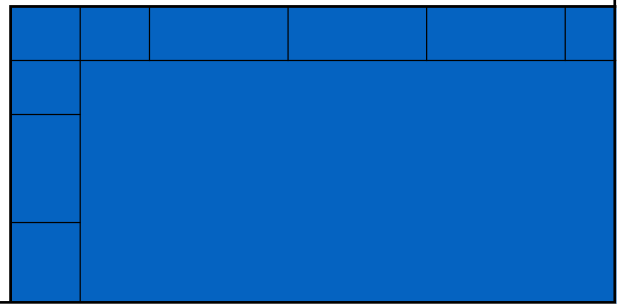
"If the table is 5 rectangles long and 6 rectangles wide, and I multiply, 5 x 6, then I have counted the corner rectangle twice."



Rectangles problem 3.

I used the turned rectangles method, and I got 32 for table A, and 22 for table B.

Can we now say something about which table is bigger?



The stick problem.

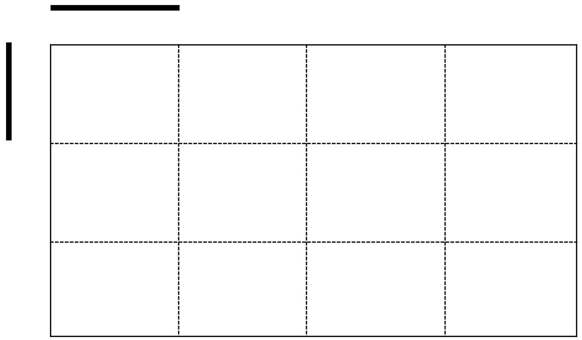
Two people work together to measure the size of a rectangular table; one measures the length and the other the width. They use a stick to measure with. The sticks, however, are of different lengths. Louisa says,

"The length is four of my sticks."

Ruiz says,

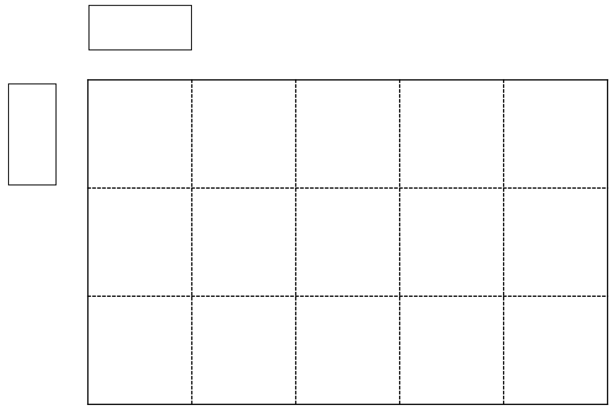
"The width is three of my sticks."

What can you say about the area of the rectangular table?



65

Original turned rectangles problem



66

anticipate, enact, analyze, reflect, revise

- Students know the area formula, but they don't understand it → hands-on activity
- Students do not seem to care about why you have to multiply; multiplying is a clever way of counting measurement units → have the students create the measurement unit: turned rectangles
- Problem too difficult; maybe carton rectangles interfere →
- Stick problem; this works, students create a measurement unit with the sticks as sides

67

Inquiry Oriented Instruction

RUME group
Inpired by RME especially Emergent Modeling

Principles

- Generating student ways of reasoning
- Building on student contributions
- Developing a shared understanding
- Connecting to standard mathematical language and notation

Kuster, Johnson, Rupnow & Wilhelm (2019).

68

Inquiry Oriented Instruction

- Practice 1. Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.
  - engaging students in cognitively demanding tasks → conjecturing, justifying,defining
- Practice 2. Teachers elicit student thinking and contributions
  - encourage students to explain their thinking and reasoningMake sense of each other's thinking
- Practice 3. Teachers actively inquire into student thinking.
  - Following up with clarification-type of questions
  - To come to understand how the students make sense of the mathematics at hand

69

Inquiry Oriented Instruction

- Practice 4. teachers are esponsive to student thinking and use student contribtions
  - Springboard for followup questions to futher progress towards the intended mathematics
  - One student's contribution → question to the entire class
- Practice 5. teachers engage students in one another's reasoning
  - Make sense of contributions of other students ⇔ revise their own thinking; shared understanding

70

Inquiry Oriented Instruction

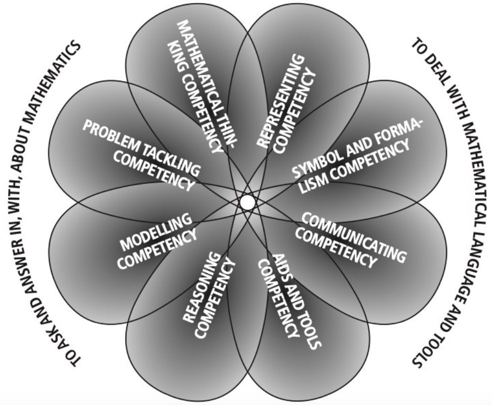
- Practice 6. Teachers guide and manage the mathematical agenda.
  - Mathematics develops from the students → which ideas to focus on (framing topics for discussion)
- Practice 7. teachers support formalizing student ideas and contributions, and introduce formal language and notation when appropriate
  - Helping students to connect their mathematics (own notation, ideas..) to that of the broader mathematics community; e.g. Make sense of textbook mathematics

71

Mandated Assessment

72

Mandated Assessment



- RME instructional design theory
  - Guided Reinvention
  - Didactical Phenomenology
  - Emergent Modeling
- Theories on teaching
  - Socio-constructivism
  - Classroom culture (social norms)
  - HLT
- Inqury Oriented Instruction
  - Principles & Practices
- Mandated assessment
  - Competencies

Thank you