

Mange tak for at invitere mig.

Jeg er meget glad og beæret over at  
være her i dag.

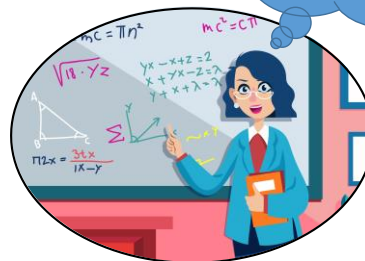
Jeg ville ønske, jeg kunne tale dansk

1

Often heard complain from many students

## Why math?

I don't need it



2

Often heard complain from many students

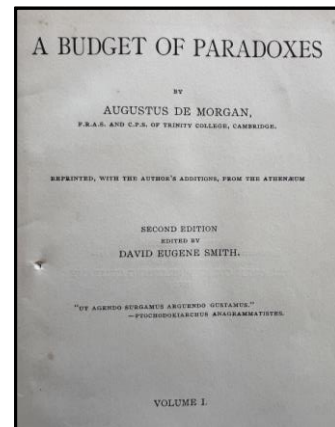
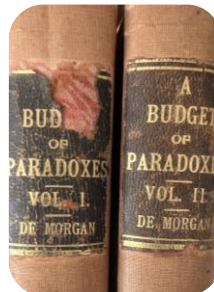
## Why math?

I don't need it

It doesn't make sense to me

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## Augustus De Morgan (1806 – 1871)



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**Denis Diderot  
(1713-1784)**

**Catherine the Great  
(1729-1796)**

**Leonhard Euler  
(1707-1783)**

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Diderot<sup>15</sup> paid a visit to the Russian Court at the invitation of the Empress. He conversed very freely, and gave the younger members of the Court circle a good deal of lively atheism. The Empress was much amused, but

6

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8

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“Monsieur!

$$\frac{a+bn}{n} = x$$

Donc Dieu existe,

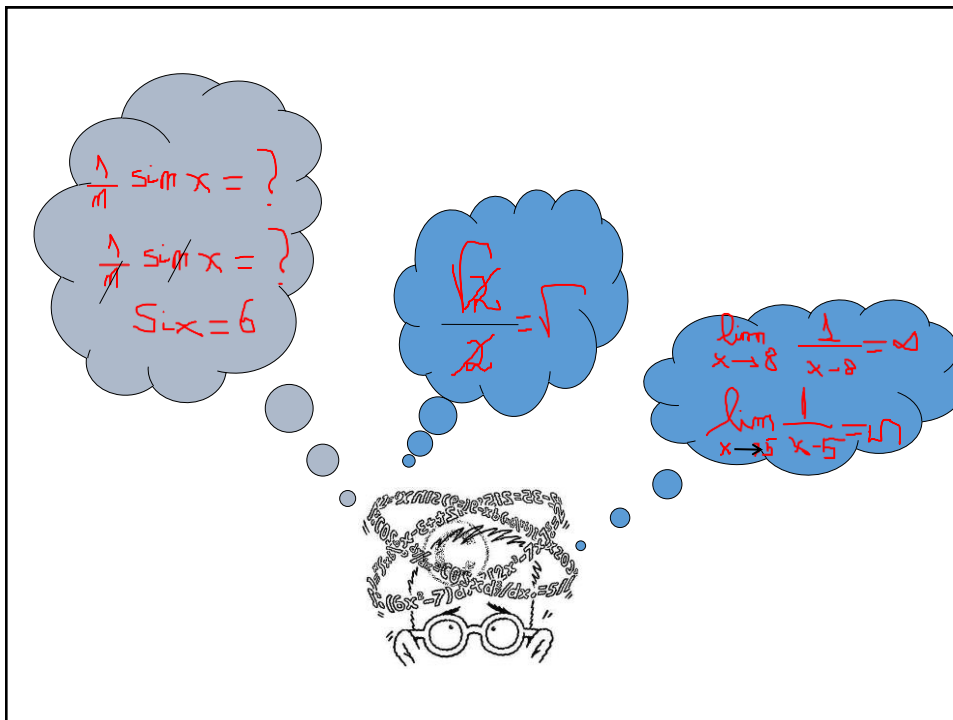
Répondez!”

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“Diderot, to whom algebra was Hebrew, was  
embarrassed and disconcerted; ...  
He asked permission to return to France at once,  
which was granted”

De Morgan, A Budget of Paradoxes  
The Open Court Publishing Co. 1915, Vol II, p.4

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AF MATEMATIKUNDERVISNING

## Mathematics and sense making

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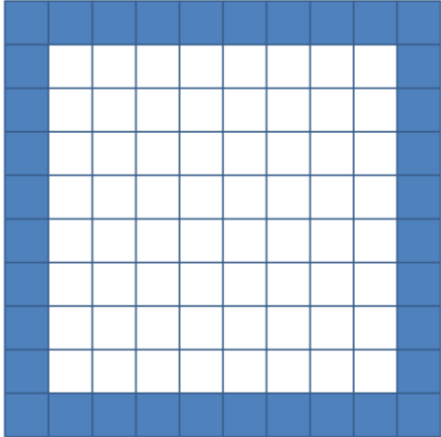
September 3, 2024

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Let's begin with "warm-up" examples

- Cells on the border of a square
- Arithmetic sequence
- String around the earth

14



$4 \times 10 - 4$   
 $4 \times 8 + 4$   
 $2 \times 10 + 2 \times 8$   
 $4 \times 9$   
 $10 \times 10 - 8 \times 8$   
 $6 \times 6$

“Situation Definition”

Boaler, J. & Humphreys C. (2005). *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning*. Heinemann.  
 Wertsch, J. V. (2011). Mediation and the zone of proximal development. In R. Miller (Ed.), *Vygotsky in Perspective*. Cambridge University Press.  
<https://doi.org/10.1017/CBO9780511736582>

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Let's begin with “warm-up” examples

- Cells on the border of a square
- Arithmetic sequence

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Given that 65 is the sum of the first ten terms of an arithmetic sequence and 20 is its tenth term, find the first term and the constant difference of the sequence.

$$S_{10} = 65$$

$$a_{10} = 20$$

$$a_1 = ?$$

$$d = ?$$

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### Resources for the solution

#### Formulas

$$S_n = na_1 + \frac{n(n-1)}{2}d$$

$$A_n = a_1 + (n-1)d$$

#### Substitution

$$\downarrow$$

$$65 = 10a_1 + 45d$$

$$20 = a_1 + 9d$$

#### Solving the equations

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? . י. .  $S_{10} = 65$

$\frac{65}{5} = 13$  : משה אחד קשה אחד משה :  $\frac{65}{5} = 13$

מכאן של המסל משה 13 ומלכיהר די-10 היל 80 האיהר  
 הילאן יהיה משה -  $\rightarrow$

הילאן יהיה משה 9 -  $\rightarrow$   $\frac{27}{9} = 3$

הילאן יהיה משה 3  $\frac{27}{9} = 3$

ההפרה :  $b = 3$

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5 arcs are 65, each arc is  $\frac{65}{5} = 13$

Since each arc is 13 and the tenth term is 20,

The first will be -7

There are 9 jumps from -7 to 20,

The distance is 27

We divide  $\frac{27}{9} = 3$       The difference is  $b=3$

Karsenty, R., Arcavi, A. & Hadas, N. (2007). Exploring informal mathematical products of low achievers at the secondary school level. *Journal of Mathematical Behavior*, 16(2), pp.156-177.

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|  |  |  |
|--|--|--|
| $S_n = na_1 + \frac{n(n-1)}{2}d$ $A_n = a_1 + (n-1)d$ <p style="text-align: center;">↓</p> $65 = 10a_1 + 45d$ $20 = a_1 + 9d$          | $S_{10} = 65$ $a_{10} = 20$ $a_1 = ?$ $d = ?$  | <p>Handwritten notes in Hebrew showing a diagram of an arithmetic sequence with terms <math>a_1, a_2, \dots, a_{10}</math> and calculations for <math>S_{10} = 65</math> and <math>a_{10} = 20</math>. The notes include the formulas <math>S_n = na_1 + \frac{n(n-1)}{2}d</math> and <math>A_n = a_1 + (n-1)d</math>, and the resulting equations <math>65 = 10a_1 + 45d</math> and <math>20 = a_1 + 9d</math>.</p> |
| <ul style="list-style-type: none"> <li>- Symbolic</li> <li>- General</li> <li>- Procedural</li> <li>- Formal justifications</li> </ul> | <ul style="list-style-type: none"> <li>- Numerical</li> <li>- Specific</li> <li>- Visual</li> <li>- Verbal</li> <li>- Transparent</li> </ul> |  |

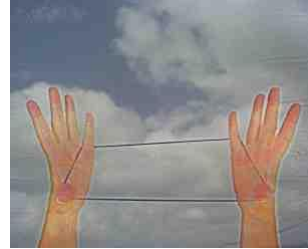
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Let's begin with "warm-up" examples

- Cells on the border of a square
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- String around the earth

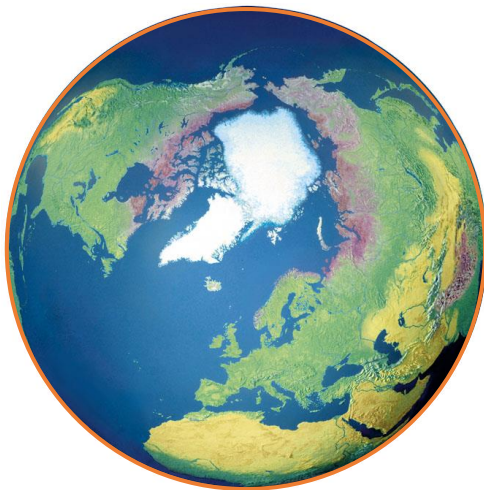
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# String around the earth



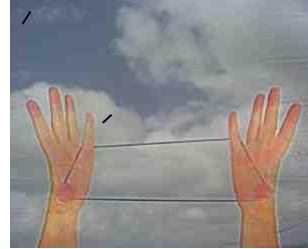
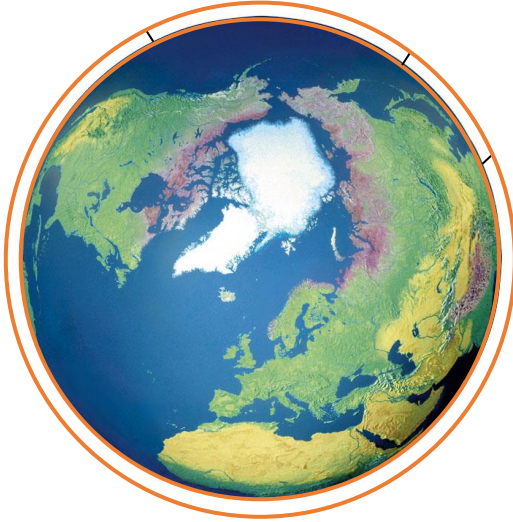
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# String around the earth



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## String around the earth

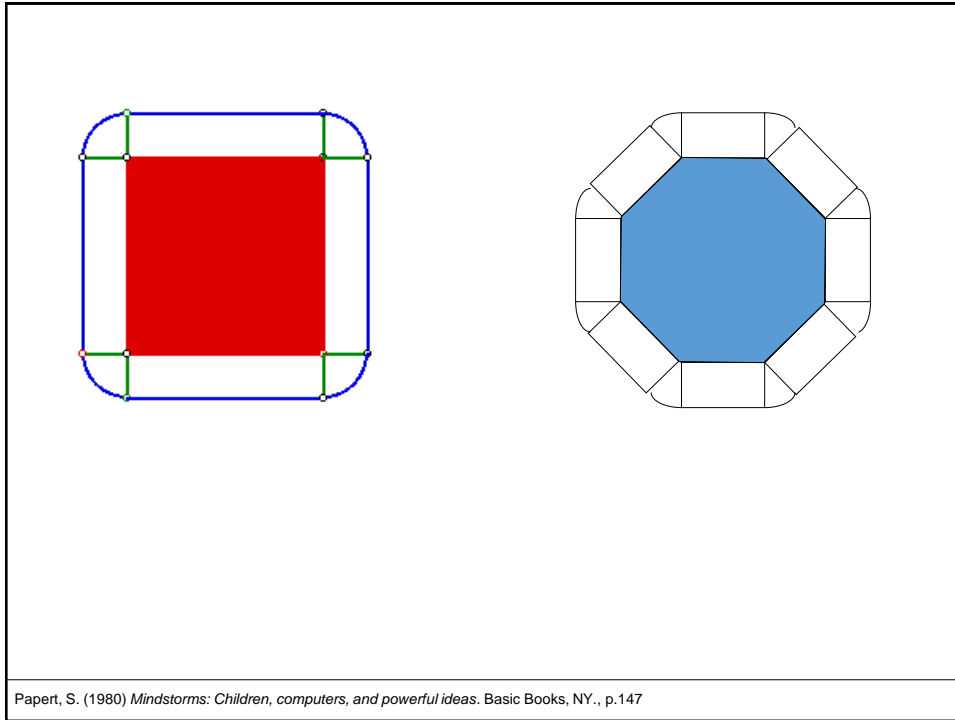


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$$2\pi(r + 2) - 2\pi r$$

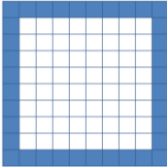
$$4\pi$$

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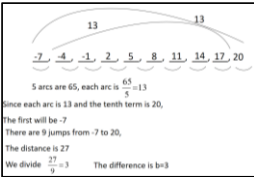


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## Mathematics and sense making





6 x 6  
"Situational"



5 arcs are 65, each arc is  $\frac{65}{5} = 13$   
 Since each arc is 13 and the tenth term is 20,  
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 There are 9 jumps from -7 to 20,  
 The distance is 27  
 We divide  $\frac{27}{3} = 9$  The difference is  $b=3$



$2\pi(r + 2) - 2\pi r$   
 $4\pi$

- Informal
- Visual
- Verbal
- Connected
- Transparency

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# Mathematics and sense making

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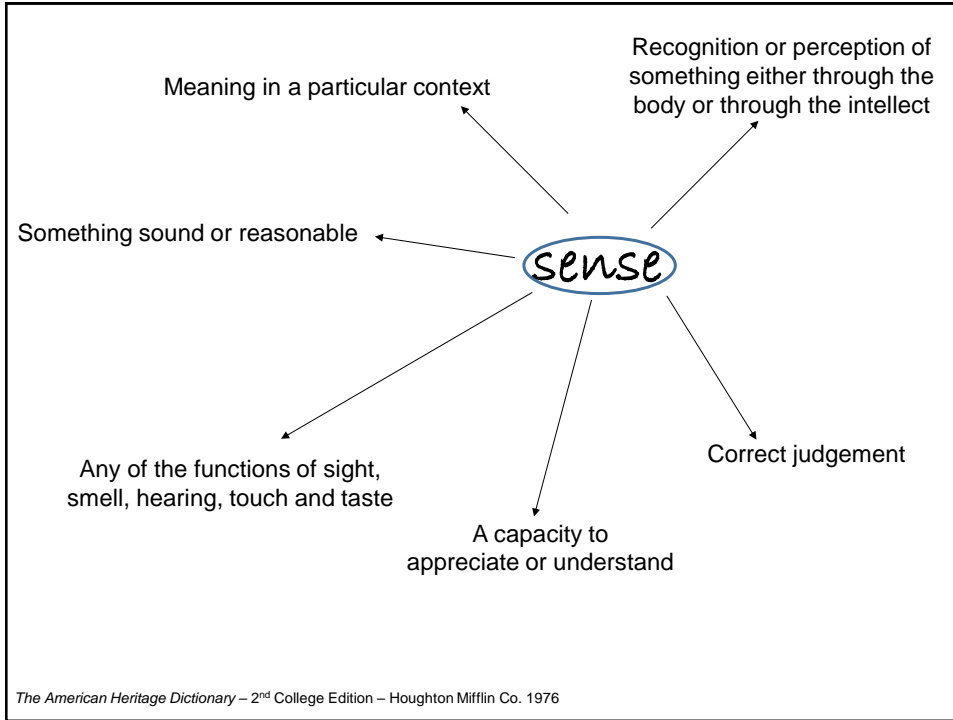
følelse

mening

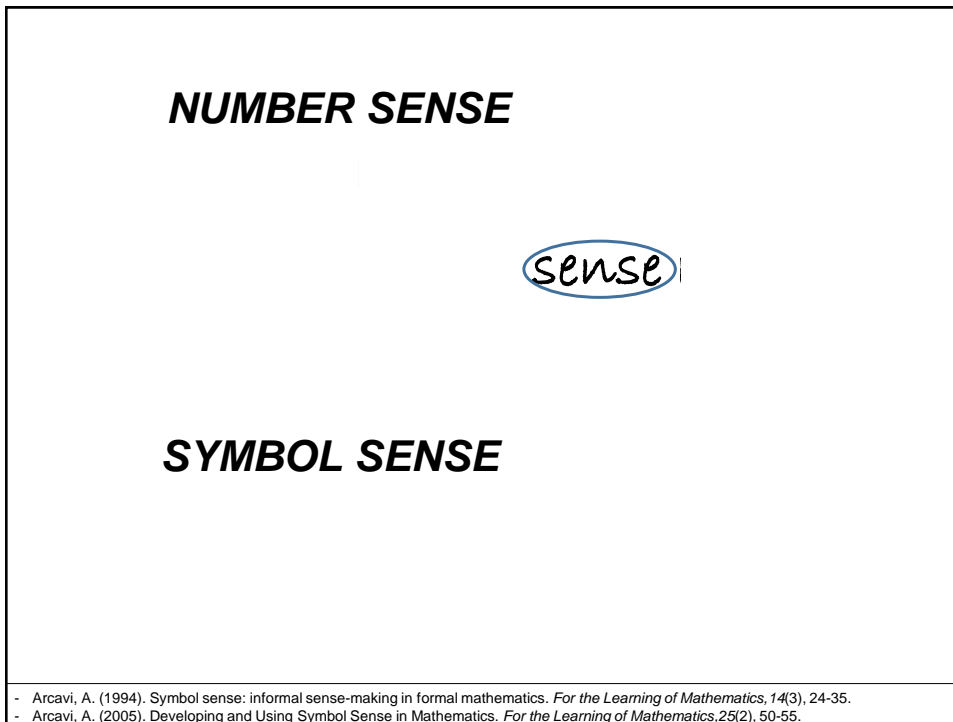
sense

sanser

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## Mathematics and sense making

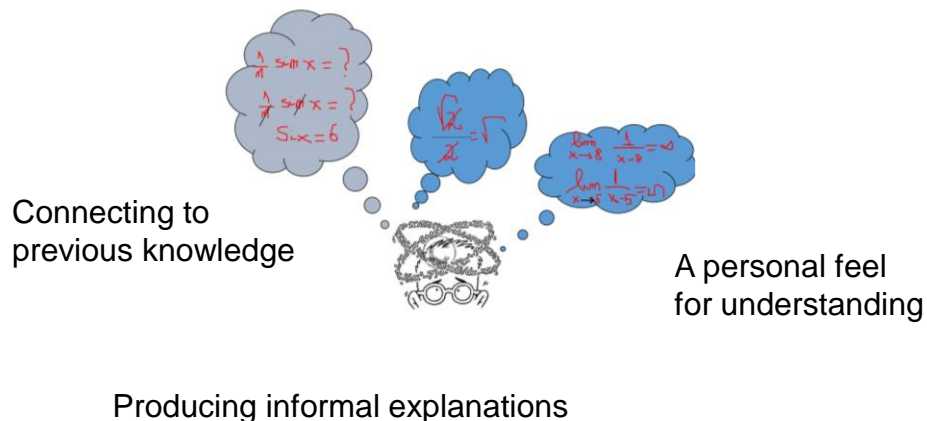
Perceiving and conceiving contexts, situations, concepts, problems, strategies and procedures by connecting them to previous knowledge and intuitions producing informal and insightful explanations and justifications which promote and sustain a personal feel for understanding.

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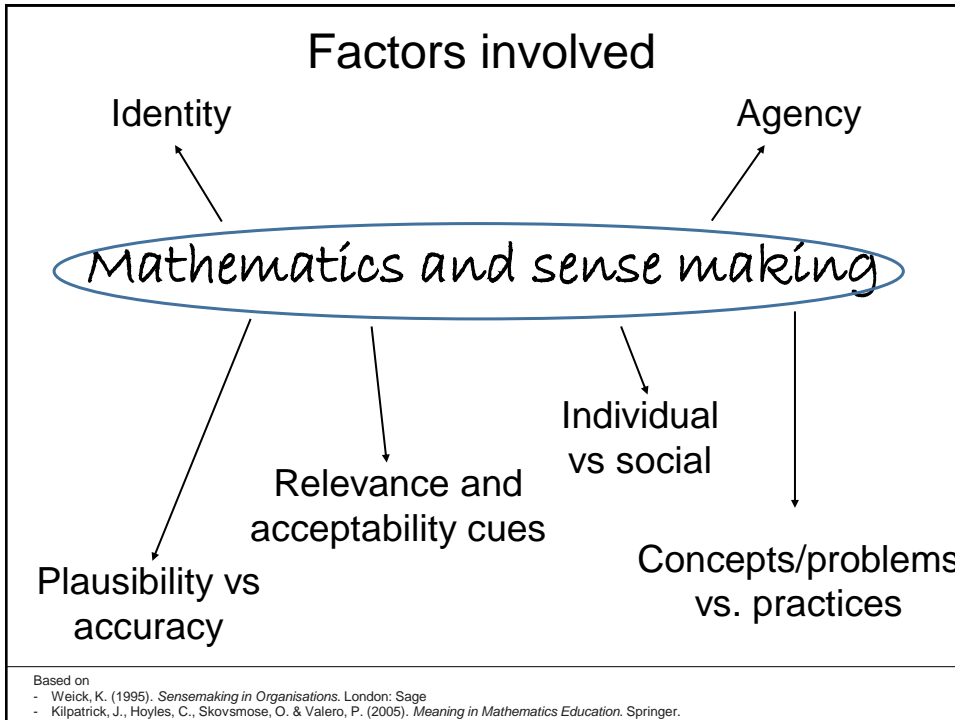
### All people are sense makers

Everybody attempts to give meaning to their experiences

Perceiving and conceiving a procedure



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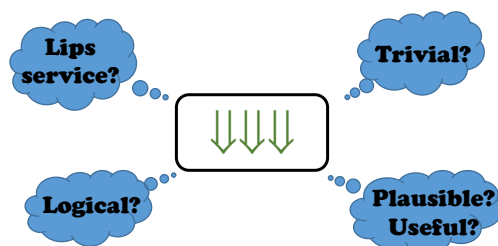
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## So, what can we do?

### Theory and Research



### Practical implications

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## So, what can we do?

### ➤ Awareness

### ➤ Listening\*

...giving careful attention to hearing what students say (and to see what they do), trying to understand it and its possible sources and entailments. ... it is not a passive undertaking...

... it should include:

- Detecting, taking up, and creating opportunities in which students are likely to engage in freely expressing their mathematical ideas;
- Questioning students to uncover the essence and sources of their ideas;
- Analyzing what one hears ... and making the enormous intellectual effort to take the 'other's perspective' in order to understand it on its own merits;
- Deciding in which ways the teaching can productively integrate students' ideas.

\* Arcavi, A., & Isoda, M. (2007). Learning to Listen: From Historical Sources to Classroom Practice. *Educational Studies in Mathematics*, 66, 111-129.  
<http://dx.doi.org/10.1007/s10649-006-9075-8>

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## **So, what can we do?**

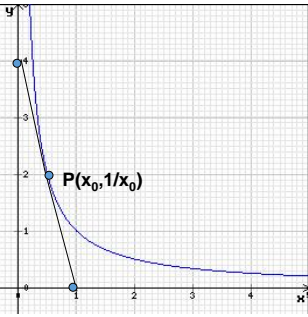
- **Awareness**
- **Listening**
- **Classroom culture, norms, values**

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**Given the function  $f(x)=1/x$ . P is a point on the graph of the function (in the first quadrant). A tangent line to the graph through P creates (with the axes) a right-angled triangle. What should the coordinates of P be, in order for the hypotenuse of that triangle to be maximum/minimum?**

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IA's solution process:



- Sketched the graph
- Found derivative of  $f(x)$ ,  $f'(x) = -1/x^2$
- Wrote equation of tangent line at P,  

$$y - y_0 = (-1/x_0^2)(x - x_0)$$
- Found coordinates of intersection with axes  
 $(2x_0, 0)$  and  $(0, 2/x_0)$
- Wrote the function for the segment length  

$$g(x) = \sqrt{4x_0^2 + \frac{4}{x_0^2}}$$
- Asked himself: max or min?

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“I have a friend who always does that [plays with the problem and makes sense of it], after such an effort, he usually has neither time nor energy to do the symbols, he does not get credit for what he may have done and fails the exams. If I don't have to, I do only the symbols, which is what the teacher and the exam want.”



Effects of deep rooted classroom practices

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## So, what can we do?

- **Awareness**
- **Listening\***
- **Classroom culture, norms, values**
- **Purposeful tasks**

43

Dakota solved the following five equations. Check her solutions, see whether they are right or wrong, and explain your answers.

$$\frac{a^2 - 9}{a^2 + 9} = 3.1$$

Friedlander, A. & Arcavi, A. (2017). *Tasks and Competencies in the Teaching and Learning of Algebra*. NCTM.

44

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"In this fraction, the numerator is always smaller than its denominator. Thus, this fraction cannot be greater than 1, and therefore this equation has no solution."

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$$\frac{a^2 - 9}{a^2 + 9} = -1$$

"A fraction is equal to  $-1$  only when its numerator and denominator are inverses (such as  $a$  and  $-a$ ). The inverse of  $a^2 - 9$  is  $9 - a^2$  and not  $a^2 + 9$ . Therefore, this equation has no solution."

Friedlander, A. & Arcavi, A. (2017). *Tasks and Competencies in the Teaching and Learning of Algebra*. NCTM.

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## So, what can we do?

- **Awareness**
- **Listening\***
- **Classroom culture, norms, values**
- **Purposeful tasks**
- **Modeling**

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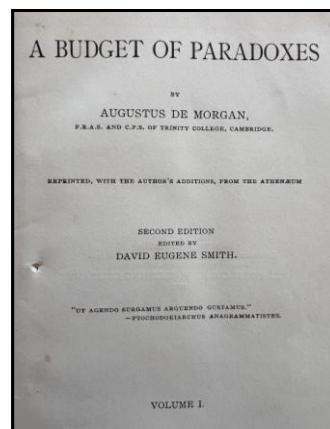
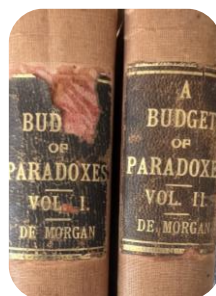


## Augustus De Morgan (1806 – 1871)



$$\frac{a+bn}{n}=x$$

Donc Dieu existe



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### NCUM | NATIONALT CENTER FOR UDVIKLING AF MATEMATIKUNDERVISNING

Mange tak for din opmærksomhed.  
Spørgsmål og kommentarer er velkomne

*Mathematics and sense making*

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September 3, 2024

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